

ELECTROMAGNETIC WAVES AND TRANSMISSION LINES

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ELECTRO MAGNETIC WAVES AND TRANSMISSION LINES

TOPICS OF UNIT-1 ELECTROSTATICS

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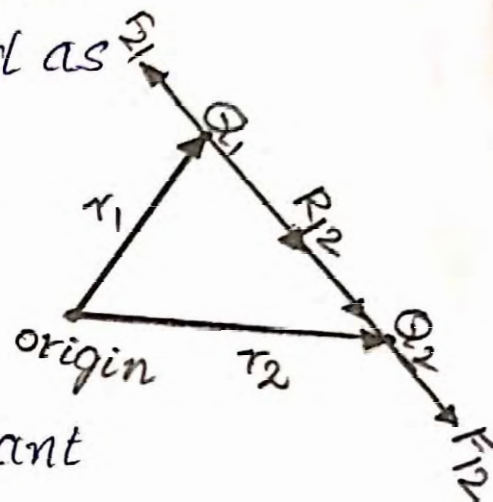
Coulomb's law:-

Coulomb's law states that the force F between two point charges Q_1 and Q_2 is

1. Along the line joining them
2. Directly proportional to the product $Q_1 Q_2$ of the charges
3. Inversely proportional to the square of the distance R between them.

It is mathematically expressed as

$$F = \frac{k Q_1 Q_2}{R^2} \rightarrow \textcircled{1}$$



where

k = proportionality constant

Q_1, Q_2 = charges in coulombs (C)

R = distance in meters (m)

F = force in Newtons (N)

$$k = \frac{1}{4\pi\epsilon_0}$$

ϵ_0 = permittivity of free space in F/m

$$\epsilon_0 = 8.854 \times 10^{-12} \approx \frac{10^{-9}}{36\pi} F/m$$

$$k = \frac{1}{4\pi\epsilon_0} \approx 9 \times 10^9 m/F$$

The equation (1) becomes.

$$\bar{F} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}$$

If point charges Q_1 and Q_2 are located at points having position vectors r_1 and r_2 then the force $\bar{F}_{12}$ on Q_2 due to Q_1 is given by

$$\boxed{\bar{F}_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} a_{R_{12}}} \rightarrow (2)$$

where

$$\bar{R}_{12} = r_2 - r_1$$

$$R = |R_{12}|$$

$$a_{R_{12}} = \frac{R_{12}}{R} \rightarrow (3)$$

substitute (3) in (2)

$$\bar{F}_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \cdot \frac{R_{12}}{R}$$

$$\boxed{\bar{F}_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^3} \cdot R_{12}} \rightarrow (4)$$

substitute R_{12} in eqⁿ (4)

$$\boxed{\vec{F}_{12} = \frac{Q_1 Q_2 (r_2 - r_1)}{4\pi\epsilon_0 |r_2 - r_1|^3}} \rightarrow (5)$$

The force F_{21} on Q_1 due to Q_2 is given by

$$\begin{aligned}\vec{F}_{21} &= |F_{12}| \vec{a}_{R_{21}} \\ &= |F_{12}| (-\vec{a}_{R_{12}})\end{aligned}$$

(or)

$$\vec{F}_{21} = -\vec{F}_{12}$$

$$\therefore \vec{a}_{R_{21}} = -\vec{a}_{R_{12}}$$

- * If we have more than two point charges, we use the principle of superposition to determine the force on a particular charge
- * The principle states that if there are N charges $Q_1, Q_2, \dots, Q_N$ located, respectively at points with position vectors $r_1, r_2, \dots, r_N$ the resultant force F on a charge Q located at point r is the vector sum of forces exerted on Q by each of charges $Q_1, Q_2, \dots, Q_N$

$$\vec{F} = \frac{Q Q_1 (r - r_1)}{4\pi\epsilon_0 |r - r_1|^3} + \frac{Q Q_2 (r - r_2)}{4\pi\epsilon_0 |r - r_2|^3} + \frac{Q Q_N (r - r_N)}{4\pi\epsilon_0 |r - r_N|^3}$$

$$\vec{F} = \frac{Q}{4\pi\epsilon_0} \sum_{k=1}^N \frac{Q_k (\vec{r} - \vec{r}_k)}{|\vec{r} - \vec{r}_k|^3} \rightarrow (6)$$

Electric field intensity:-

- * It is also known as electric field strength
- * The electric field intensity (E) is the force per unit charge when placed in an electric field.

$$\vec{E} = \frac{\vec{F}}{Q}$$

For $Q > 0$, the electric field intensity E is obviously in the direction of the force F and is measured in newtons per coulomb or volts per meter.

The electric field intensity at point r due to a point charge located at r' is readily obtained

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 R^2} \vec{a}_R$$

$$\vec{E} = \frac{Q (\vec{r} - \vec{r}')}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|^3}$$

$$(\because \vec{a}_R = \frac{\vec{R}}{|\vec{R}|})$$

For N point charges $Q_1, Q_2, \dots, Q_N$ located at $r_1, r_2, \dots, r_N$ the electric field intensity at point r is obtained

$$\bar{E} = \frac{Q_1(r-r_1)}{4\pi\epsilon_0|r-r_1|^3} + \frac{Q_2(r-r_2)}{4\pi\epsilon_0|r-r_2|^3} + \dots + \frac{Q_N(r-r_N)}{4\pi\epsilon_0|r-r_N|^3}$$

Therefore,

$$\boxed{\bar{E} = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^N \frac{Q_k(r-r_k)}{|r-r_k|^3}}$$

problem:-

- * point charges 1mc and -2mc are located at $(3, 2, -1)$ and $(-1, -1, 4)$ respectively. calculate the electric force on a 10nc charge located at $(0, 3, 1)$ and the electric field intensity at that point.

sol:- we know that

$$\bar{F} = \sum_{k=1,2} \frac{Q Q_k}{4\pi\epsilon_0 R^2} \bar{a}_R = \sum_{k=1,2} \frac{Q Q_k (r-r_k)}{4\pi\epsilon_0 |r-r_k|^3}$$

simply,

$$\bar{F}_{12} = \frac{Q Q_1 (r-r_1)}{4\pi\epsilon_0 |r-r_1|^3} + \frac{Q Q_2 (r-r_2)}{4\pi\epsilon_0 |r-r_2|^3}$$

$$\vec{F}_{12} = \frac{Q}{4\pi\epsilon_0} \left[\frac{Q_1(r-r_1)}{|r-r_1|^3} + \frac{Q_2(r-r_2)}{|r-r_2|^3} \right]$$

$$\vec{F}_{12} = \frac{10 \times 10^{-9}}{4\pi \left(\frac{10^{-9}}{36\pi} \right)} \left[\frac{10^{-3} [(0,3,1) - (3,2,1)]}{|(0,3,1) - (3,2,1)|^3} - \frac{2 \times 10^{-3} [(0,3,1) - (-1,-1,4)]}{|(0,3,1) - (-1,-1,4)|^3} \right]$$

$$\vec{F}_{12} = 10 \times 10^{-9} \times 9 \times 10^9 \left[\frac{10^{-3} (-3,1,2)}{(\sqrt{9+1+4})^3} - \frac{2(1,4,-3)}{(\sqrt{1+16+9})^3} \right]$$

$$\vec{F}_{12} = 9 \times 10^1 \times 10^{-3} \left[\frac{(-3,1,2)}{14\sqrt{14}} + \frac{(-2,-8,6)}{26\sqrt{26}} \right]$$

$$\vec{F}_{12} = 9 \times 10^{-2} \left[\frac{(-3,1,2)}{14\sqrt{14}} + \frac{(-2,-8,6)}{26\sqrt{26}} \right]$$

$$\vec{F}_{12} = -6.507 a_x - 3.817 a_y + 7.506 a_z \text{ mN}$$

At effct point,

$$\vec{E} = \frac{\vec{F}}{Q}$$

$$\vec{E} = (-6.507, -3.817, 7.506) \cdot \frac{10^{-3}}{10 \cdot 10^{-9}}$$

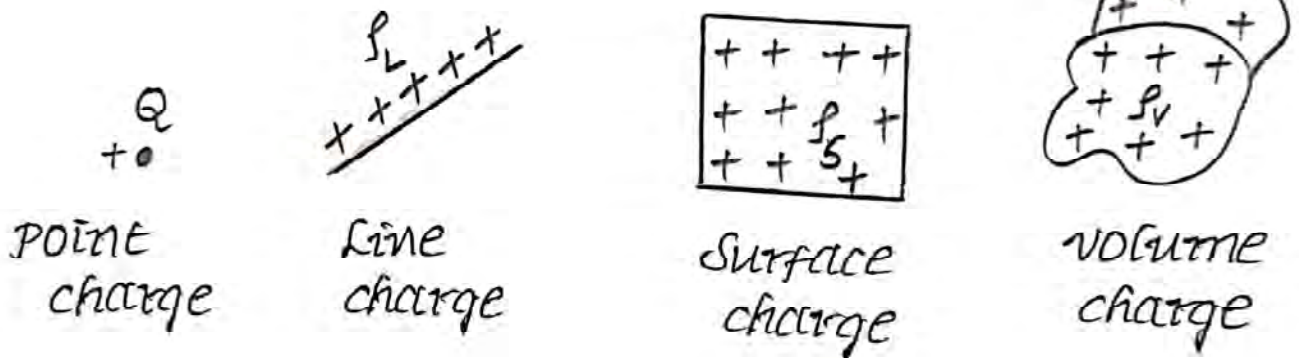
$$\vec{E} = -650.7 a_x - 381.7 a_y + 750.6 a_z \text{ KV/m}$$

Electric fields due to continuous charge distributions:-

The charge element dQ and the total charge Q due to these charge distributions are obtained.

These are of four types

1. point charge
2. Line charge
3. Surface charge
4. Volume charge



Line charge - $dQ = \rho_L dL$

$$Q = \int_L \rho_L dL$$

Surface charge - $dQ = \rho_S ds$

$$Q = \int_S \rho_S ds$$

Volume charge - $dQ = \rho_V dv$

$$Q = \int_V \rho_V dv$$

The electric field intensity due to each of the charge distributions ρ_L, ρ_S and ρ_V may be regarded as the summation of the field contributed by numerous point charges making up the charge distribution.

$$\therefore \bar{E} = \int_L \frac{\rho_L dL}{4\pi\epsilon_0 R^2} \mathbf{a}_R \rightarrow \text{line charge}$$

$$\bar{E} = \int_S \frac{\rho_S dS}{4\pi\epsilon_0 R^2} \mathbf{a}_R \rightarrow \text{surface charge}$$

$$\bar{E} = \int_V \frac{\rho_V dV}{4\pi\epsilon_0 R^2} \mathbf{a}_R \rightarrow \text{volume charge}$$

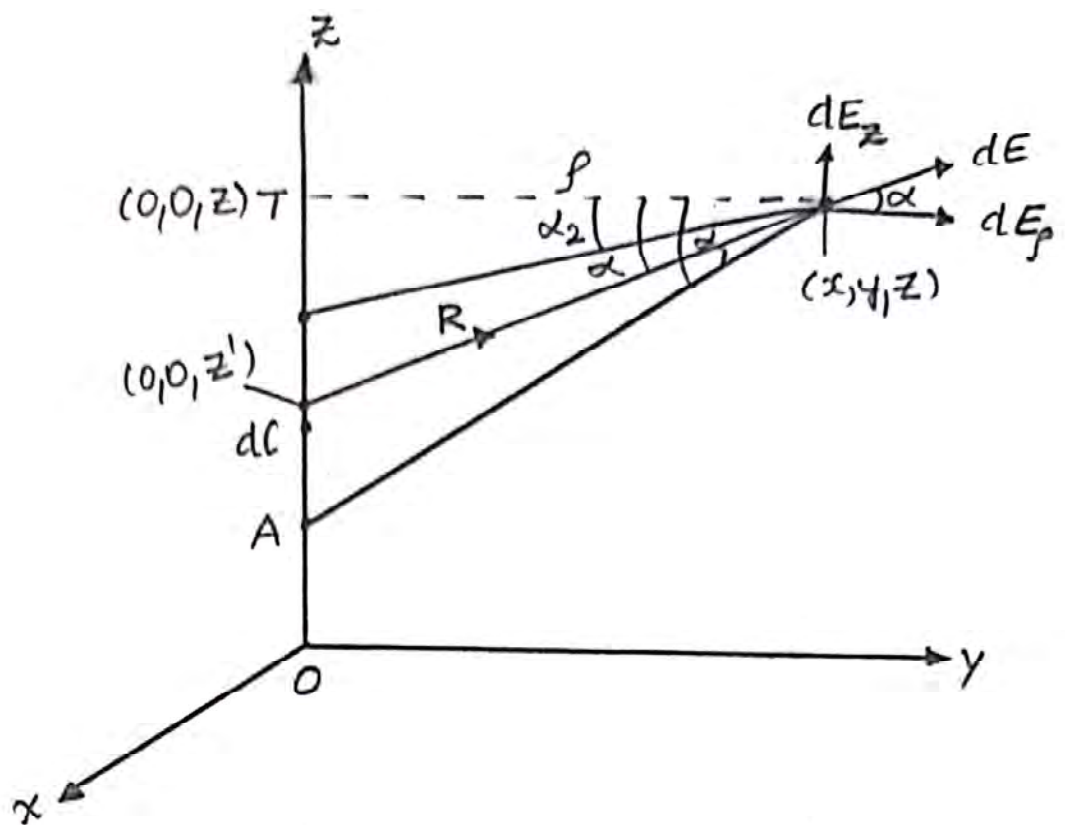
Line charge:-

consider a line charge with uniform charge density ρ_L extending from A to B along the z-axis. The charge element dQ associated with element $dL = dz$ of the line is

$$dQ = \rho_L dL = \rho_L dz$$

Hence the total charge Q is

$$Q = \int_{z_A}^{z_B} \rho_L dz$$



The electric field intensity E at an arbitrary point $P(x, y, z)$ can be found by using equation

$$\vec{E} = \int_L \frac{\rho_L dL}{4\pi\epsilon_0 R^2} \vec{a}_R \rightarrow (1)$$

It is customary to denote the field point by (x, y, z) and the source point by (x', y', z') .

$$dL = dz'$$

$$\vec{R} = (x, y, z) - (0, 0, z')$$

$$\vec{R} = x\vec{a}_x + y\vec{a}_y + (z - z')\vec{a}_z$$

(or)

$$\vec{R} = \rho\vec{a}_\rho + (z - z')\vec{a}_z$$

$$\begin{aligned} R^2 &= |\vec{R}|^2 = x^2 + y^2 + (z - z')^2 \\ &= \rho^2 + (z - z')^2 \end{aligned}$$

$$\frac{\vec{a}_R}{R^2} = \frac{\vec{R}}{|\vec{R}|^3} = \frac{\rho \vec{a}_\rho + (z - z') \vec{a}_z}{[\rho^2 + (z - z')^2]^{3/2}}$$

Substituting all into equation (1)

$$\vec{E} = \frac{\rho_L}{4\pi\epsilon_0} \int \frac{\rho \vec{a}_\rho + (z - z') \vec{a}_z}{[\rho^2 + (z - z')^2]^{3/2}} dz' \rightarrow (2)$$

To evaluate this, it is convenient that we define α_1, α_1 and α_2 as shown in the figure

From the above diagram,

$$\tan \alpha = \frac{z - z'}{\rho}$$

$$\sec \alpha = \frac{R}{\rho}$$

$$z - z' = \rho \tan \alpha$$

$$R = \rho \sec \alpha$$

$$z' = z - \rho \tan \alpha$$

$$dz' = 0 - \rho \sec^2 \alpha d\alpha$$

$$dz' = -\rho \sec^2 \alpha d\alpha$$

$$\therefore R = [\rho^2 + (z - z')^2]^{1/2} = \rho \sec \alpha$$

$$z' = 0 - \rho \tan \alpha$$

$$dz' = -\rho \sec^2 \alpha d\alpha$$

$\therefore$ The eqn (2) becomes

$$\vec{E} = \frac{-\rho_L}{4\pi\epsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{\rho \sec^2 \alpha [\cos \alpha \vec{a}_\rho + \sin \alpha \vec{a}_z]}{\rho^2 \sec^2 \alpha} d\alpha$$

$$\vec{E} = -\frac{\rho_L}{4\pi\epsilon_0\rho} \int_{\alpha_1}^{\alpha_2} [\cos\alpha \vec{a}_\rho + \sin\alpha \vec{a}_z] d\alpha$$

Thus for a finite line charge,

$$\vec{E} = -\frac{\rho_L}{4\pi\epsilon_0\rho} \left[[\sin\alpha]_{\alpha_1}^{\alpha_2} \vec{a}_\rho - [\cos\alpha]_{\alpha_1}^{\alpha_2} \vec{a}_z \right]$$

$$\vec{E} = \frac{\rho_L}{4\pi\epsilon_0\rho} \left[[\cos\alpha_2 - \cos\alpha_1] \vec{a}_z - [\sin\alpha_2 - \sin\alpha_1] \vec{a}_\rho \right]$$

For infinite line charge, considering special case,

point B is at $(0,0,\infty)$ and A at $(0,0,-\infty)$

so that $\alpha_1 = \frac{\pi}{2}$, $\alpha_2 = -\frac{\pi}{2}$; the z-component

vanishes and therefore the equation is

$$\vec{E} = \frac{\rho_L}{4\pi\epsilon_0\rho} \left[\left(\cos\left(-\frac{\pi}{2}\right) - \cos\frac{\pi}{2} \right) \vec{a}_z - \left(\sin\left(-\frac{\pi}{2}\right) - \sin\frac{\pi}{2} \right) \vec{a}_\rho \right]$$

$$\vec{E} = \frac{\rho_L}{4\pi\epsilon_0\rho} \left[(0-0)\vec{a}_z - (-1-1)\vec{a}_\rho \right]$$

$$\vec{E} = \frac{\rho_L}{4\pi\epsilon_0\rho} [2\vec{a}_\rho]$$

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon_0 r} \cdot \vec{a}_r$$

As the above equation is obtained for an infinite line charge along the z-axis so that ρ and a_r have their usual meaning.

If the line is not along the z-axis, ρ is the perpendicular distance from the line to the point of interest and a_r is a unit vector along that distance directed from the line charge to the field point.

Surface charge:-

consider an infinite sheet of charge in the xy-plane with uniform charge density ρ_s .

The charge associated with an elemental area ds is

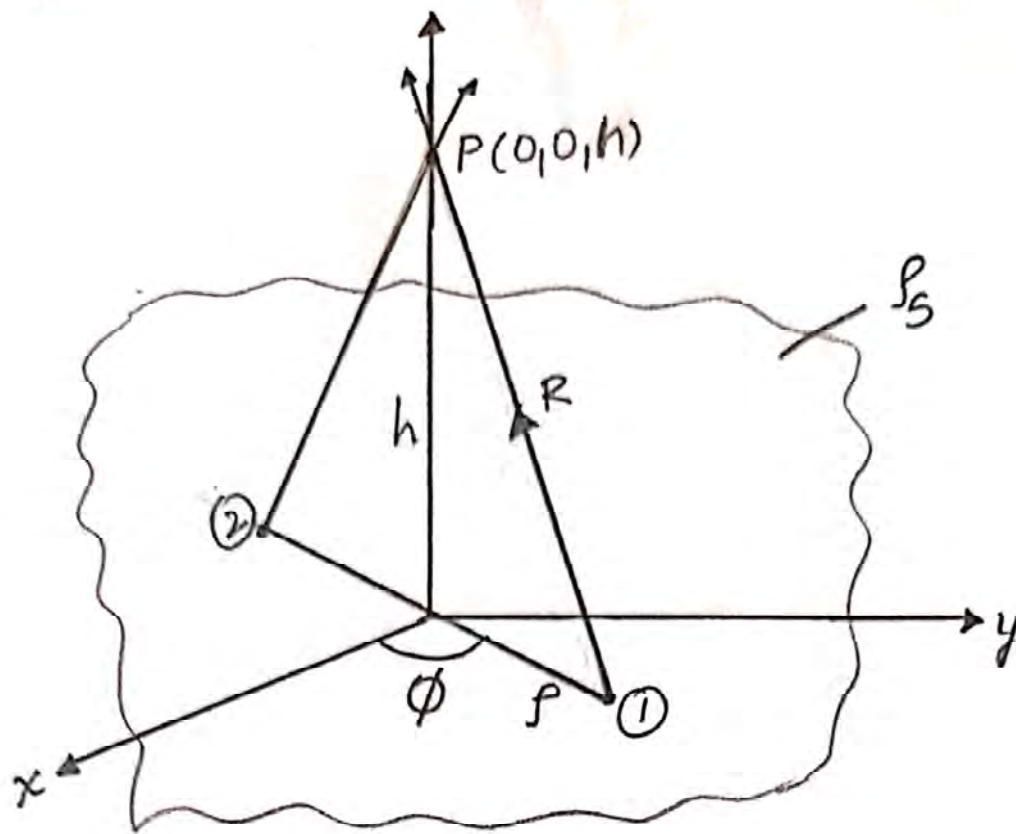
$$dQ = \rho_s ds$$

From the equation,

$$\vec{E} = \int_S \frac{\rho_s ds}{4\pi\epsilon_0 R^2} \vec{a}_R$$

The contribution to the E field at point $P(0,0,h)$ by the charge dQ on the elemental surface is

$$d\vec{E} = \frac{dQ}{4\pi\epsilon_0 R^2} \vec{a}_R \rightarrow \textcircled{1}$$



From the above figure,

$$\vec{R} = \rho(-\hat{a}_\rho) + h\hat{a}_z$$

$$R = |\vec{R}| = [\rho^2 + h^2]^{1/2}$$

$$\vec{a}_R = \frac{\vec{R}}{R}$$

$$dQ = \rho_S ds = \rho_S \rho d\phi d\rho$$

substituting in eqn ①

$$d\vec{E} = \frac{\rho_S \rho d\phi d\rho [-\rho \hat{a}_\rho + h\hat{a}_z]}{4\pi\epsilon_0 [\rho^2 + h^2]^{3/2}}$$

The contributions to \$E_\rho\$ add up to zero so that \$E\$ has only \$z\$-component. This can be shown mathematically by replacing \$\hat{a}_\rho\$ with

$\cos\phi a_x + \sin\phi a_y$. Integration of $\cos\phi$ or $\sin\phi$ over $0 < \phi < 2\pi$ gives zero.

Therefore,

$$\vec{E} = \int_S d\vec{E}_z$$

$$\vec{E} = \frac{\rho_s}{4\pi\epsilon_0} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\infty} \frac{h\rho d\rho d\phi}{[\rho^2+h^2]^{3/2}} \vec{a}_z$$

$$\vec{E} = \frac{\rho_s h}{4\pi\epsilon_0} (2\pi) \int_0^{\infty} [\rho^2+h^2]^{-3/2} \cdot \frac{1}{2} d(\rho^2) \vec{a}_z$$

$$\vec{E} = \frac{\rho_s h}{2\epsilon_0} \left[\left\{ -[\rho^2+h^2]^{-1/2} \right\} \right]_0^{\infty} \vec{a}_z$$

$$\boxed{\vec{E} = \frac{\rho_s}{2\epsilon_0} \vec{a}_z} \rightarrow (2)$$

E has only z -component if the charge is in the xy -plane. eqⁿ (2) is valid for $h > 0$; for $h < 0$ we would need to replace $\vec{a}_z$ with $-\vec{a}_z$. In general, for an infinite sheet of charge

$$\boxed{\vec{E} = \frac{\rho_s}{2\epsilon_0} \vec{a}_n} \rightarrow (3)$$

where $\vec{a}_n$ is a unit vector normal to the sheet. From eqⁿs (2) and (3) we notice that the electric field is normal to the sheet and it is surprisingly independent of the distance between the sheet and the point of observation P .

In a parallel plate - capacitor, the electric field existing between the two plates having equal and opposite charges is given by

$$\vec{E} = \frac{\rho_s}{2\epsilon_0} \vec{a}_n + \left(\frac{-\rho_s}{2\epsilon_0} \right) (-\vec{a}_n)$$

$$\boxed{\vec{E} = \frac{\rho_s}{\epsilon_0} \vec{a}_n}$$

Volume charge:-

Let us consider a sphere of radius a centered at the origin. Let the volume of the sphere be filled uniformly with a volume charge density ρ_v (in C/m^3). The charge associated with elemental volume dv is

$$dQ = \rho_v dv$$

∴ The equation for volume charge is

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \vec{a}_r$$

electric flux density:-

The flux due to electric field E can be calculated by using the general definition of flux.

The electric field intensity is dependent on the medium in which the charge is placed (free space). Suppose a new vector field D is defined by

$$\vec{D} = \epsilon_0 \vec{E}$$

We use line charge equation to define electric flux ψ in terms of D , namely

$$\psi = \int_S \vec{D} \cdot d\vec{s}$$

For an infinite sheet of charge

$$\vec{D} = \frac{\rho_s}{2} \vec{a}_n$$

For volume charge distribution,

$$\vec{D} = \int_V \frac{\rho_v a_v}{4\pi R^2} \vec{a}_R$$

Divergence theorem:-

The divergence theorem states that the outward flux of an electric field $\vec{A}$ through the closed surface is same as the volume integral of the divergence of that vector field. ($\vec{A}$)

$$\int_S \vec{A} \cdot d\vec{s} = \oint_V \nabla \cdot \vec{A} dv$$

$$\iint_S \vec{A} \cdot d\vec{s} = \iiint_V \nabla \cdot \vec{A} dv$$

STOKES theorem:-

The STOKES theorem states that the vector field $\vec{A}$ with the length is equal to the surface integral of curl of $\vec{A}$
(or)

It states that the circulation of a vector field A around a (closed) path L is equal to the surface integral of curl of A over the open surface S bounded by L , provided A and $\nabla \times A$ are continuous on S .

$$\int \vec{A} \cdot d\vec{C} = \iint \nabla \times \vec{A} d\vec{s}$$

Gauss Law:-

Gauss law states that the total electric flux ψ through any closed surface is equal to the total charge enclosed by that surface

$$\boxed{\psi = Q_{enc}}$$

That is,

$$\psi = \oint_S d\psi = \oint_S \vec{D} \cdot d\vec{s}$$

total charge enclosed $Q = \int_V \rho_v dv$

From the definition,

$$\psi = Q$$

$$\therefore \boxed{Q = \oint_S \vec{D} \cdot d\vec{s} = \int_V \rho_v dv} \rightarrow \textcircled{1}$$

By applying divergence theorem,

$$\oint_S \vec{D} \cdot d\vec{s} = \int_V \nabla \cdot \vec{D} \cdot dv \rightarrow \textcircled{2}$$

substituting $\textcircled{2}$ in $\textcircled{1}$

$$\iint \nabla \cdot \vec{D} dv = \iiint \rho_v dv$$

$$\iiint (\nabla \cdot \vec{D} - \rho_v) dv = 0$$

$$\boxed{\nabla \cdot \vec{D} = \rho_v} \rightarrow \textcircled{3}$$

which is the first of four Maxwell's equations to be derived.

eqⁿ ③ states that the volume charge density is the same as the divergence of the electric flux density.

Applications of Gauss law:-

The procedure for applying Gauss's law to calculate the electric field involves first knowing whether symmetry exists.

It is applicable to following cases.

1. Point charge:-

Suppose a point charge Q is located at the origin. To determine D at a point P , it is easy to see that choosing a spherical surface containing P will satisfy symmetry conditions. Thus, a spherical surface centered at the origin is the Gaussian surface

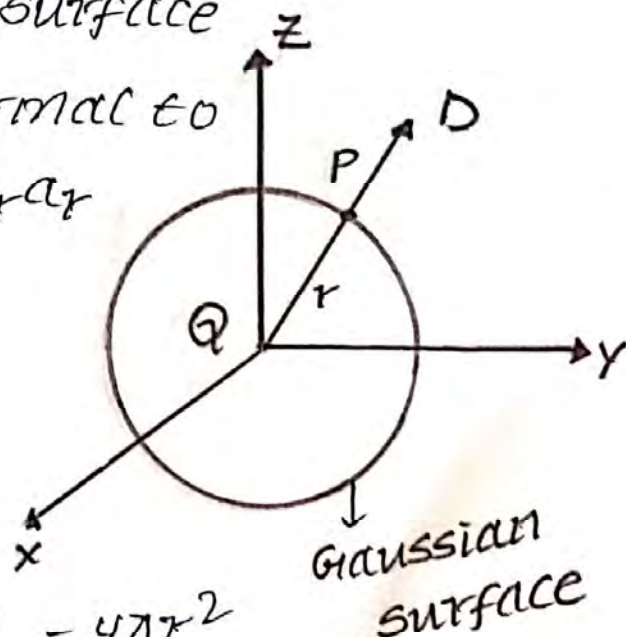
since D is everywhere normal to Gaussian surface, i.e., $D = D_r a_r$

Applying Gauss law,

$$Q = \oint_S \bar{D} \cdot d\bar{s} = \bar{D}_r \oint_S ds$$

$$Q = \bar{D}_r 4\pi r^2$$

$$\oint ds = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta d\theta d\phi = 4\pi r^2$$



is the surface area of Gaussian surface

Thus

$$\bar{D} = \frac{Q}{4\pi r^2} \mathbf{a}_r$$

2. Infinite line charge:-

Suppose the infinite line of uniform charge ρ_L C/m lies along the z-axis. The electric flux density D is constant on and normal to the cylindrical gaussian surface i.e.,

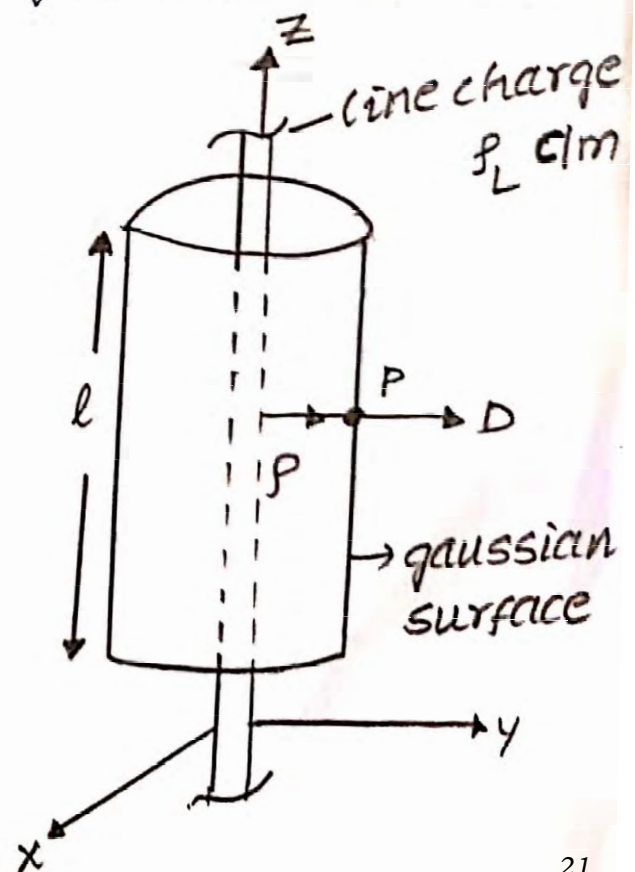
$D = D_\rho \mathbf{a}_\rho$. If we apply gauss's law to an arbitrary length l of line

$$\rho_L l = Q = \oint_S \bar{D} \cdot d\mathbf{s} = \bar{D}_\rho \oint_S d\mathbf{s} = \bar{D}_\rho 2\pi \rho l$$

where $\oint d\mathbf{s} = 2\pi \rho l$ is gaussian surface.

Therefore,

$$\bar{D} = \frac{\rho_L}{2\pi \rho} \mathbf{a}_\rho$$



3. Infinite sheet of charge:-

The expression for infinite sheet of charge is

$$\boxed{\bar{D} = \frac{\rho_s}{2} \bar{a}_z}$$

(091)

$$\boxed{\bar{E} = \frac{\bar{D}}{\epsilon_0} = \frac{\rho_s}{2\epsilon_0} \bar{a}_z}$$

4. uniformly charged sphere:-

The expression for uniformly charged sphere is

$$\bar{D} = \begin{cases} \frac{r}{3} \rho_0 \bar{a}_r, & 0 < r \leq a \\ \frac{a^3}{3r^2} \rho_0 \bar{a}_r, & r \geq a \end{cases}$$

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Electric potential:-

Suppose we wish to move a point charge Q from point A to point B in an electric field E .

From coulomb's law,

$$\vec{F} = Q\vec{E}$$

The work done in displacing charge $d\ell$ is

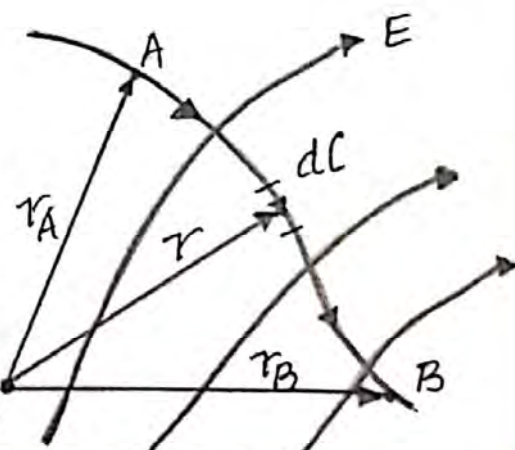
$$dW = -\vec{F} \cdot d\vec{\ell}$$

$$dW = -Q\vec{E} \cdot d\vec{\ell}$$

The negative sign indicates that the work is being done by an external agent. Thus the total work done, or the potential energy required, in moving Q from A to B is.

$$W = -Q \int_A^B \vec{E} \cdot d\vec{\ell}$$

Dividing $\frac{W}{Q}$ in the above eqⁿ gives the potential energy per unit charge. This quantity, denoted by V_{AB} is known as potential difference between points A and B. Thus



$$\vec{V}_{AB} = \frac{W}{Q} = - \int_A^B \vec{E} \cdot d\vec{\ell} \rightarrow \textcircled{1}$$

If the E field in figure is due to a point charge Q located at the origin, then

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{a}_r$$

The eqⁿ ① becomes

$$V_{AB} = - \int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2} \hat{a}_r \cdot d\vec{r} \hat{a}_r$$

$$dC = \hat{a}_r \cdot d\vec{r}$$

$$V_{AB} = - \int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2} dr$$

$$V_{AB} = - \frac{Q}{4\pi\epsilon_0} \left[\frac{-1}{r} \right]_{r_A}^{r_B}$$

$$V_{AB} = \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r} \right]_{r_A}^{r_B}$$

$$V_{AB} = \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r_B} - \frac{1}{r_A} \right]$$

$$V_{AB} = \frac{Q}{4\pi\epsilon_0 r_B} - \frac{Q}{4\pi\epsilon_0 r_A}$$

$$\boxed{V_{AB} = V_B - V_A}$$

The potential at any point is the potential difference between the point and a chosen point (or reference point) at which the potential is zero.

$$V_{AB} = - \int_A^B \vec{E} \cdot d\vec{C} + C$$

$$V_{AB} = \frac{Q}{4\pi\epsilon_0 r} + C$$

$$V_{AB} = -V_{BA}$$

$$V_{AB} + V_{BA} = 0$$

$$\boxed{V = 0}$$

Problem:-

- * Two point charges $-4\mu C$ and $5\mu C$ are located at $(2, -1, 3)$ and $(0, 4, -2)$ respectively. Find the potential at $(1, 0, 1)$ assuming zero potential at infinity.

Solr Let $Q_1 = -4\mu C$, $Q_2 = 5\mu C$

$$V(r) = \frac{Q_1}{4\pi\epsilon_0 |r-r_1|} + \frac{Q_2}{4\pi\epsilon_0 |r-r_2|} + C_0$$

$$\text{If } V(\infty) = 0, C_0 = 0$$

$$|r-r_1| = |(1, 0, 1) - (2, -1, 3)| = |(-1, 1, -2)| = \sqrt{6}$$

$$|r-r_2| = |(1, 0, 1) - (0, 4, -2)| = |(1, -4, 3)| = \sqrt{26}$$

Hence

$$\begin{aligned} V(1, 0, 1) &= \frac{10^{-6}}{4\pi \times 10^{-9} \times 36\pi} \left[\frac{-4}{\sqrt{6}} + \frac{5}{\sqrt{26}} \right] \\ &= 9 \times 10^3 (-1.633 + 0.9806) \\ &= -5.872 \text{ kV} \end{aligned}$$

Relationship between E and V :

The potential difference between points A and B is independent of the path taken. Hence,

$$V_{BA} = -V_{AB}$$

$$\text{i.e., } V_{BA} + V_{AB} = \oint E \cdot dC = 0$$

$$\boxed{\oint_L \vec{E} \cdot dC = 0}$$

Applying STOKES'S THEOREM,

$$\oint_L \vec{E} \cdot dC = \int_S (\nabla \times \vec{E}) \cdot dS = 0$$

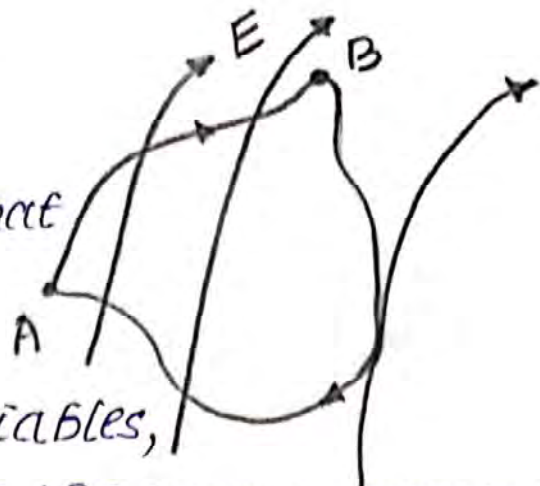
$$\boxed{\nabla \times \vec{E} = 0} \rightarrow \textcircled{1}$$

Thus an electrostatic field is a conservative field. The eqⁿ $\textcircled{1}$ is referred to Maxwell's equation for static electric fields.

From the way we defined potential,

$V = - \int \vec{E} \cdot d\vec{l}$ it follows that

$$dV = -\vec{E}_x dx - \vec{E}_y dy - \vec{E}_z dz$$



From calculus of multivariables, a total change in $V(x, y, z)$ is the sum of partial changes with respect to x, y, z variables.

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

comparing two expressions for dV , we obtain

$$\vec{E}_x = -\frac{\partial V}{\partial x}, \quad \vec{E}_y = -\frac{\partial V}{\partial y}, \quad \vec{E}_z = -\frac{\partial V}{\partial z}$$

Thus

$$\boxed{\vec{E} = -\nabla V}$$

That is, the electric field intensity is the gradient of V . The negative sign shows that the direction of E is opposite to the direction in which V increases; E is directed from higher to lower levels of V .

Maxwell's equations:-

$$\nabla \cdot \vec{D} = \rho_v$$

$$\nabla \times \vec{E} = 0$$

Energy Density in electrostatic Fields:-

Suppose we wish to position three point charges Q_1, Q_2 and Q_3 in an initially empty space

1. NO work is required to transfer Q_1 from infinity to P_1 because the space is initially charge free and there is no electric field.
[$W=0$]
2. The work done in transferring Q_2 from infinity to P_2 is equal to the product of Q_2 and the potential V_{21} at P_2 due to Q_1 .
3. The work done in positioning Q_3 at P_3 is equal to $Q_3 (V_{32} + V_{31})$ where V_{32} and V_{31} are the potentials at P_3 due to Q_2 and Q_1 respectively.

$\therefore$ The total work done in positioning three charges is

$$W_E = W_1 + W_2 + W_3$$

$$W_E = 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32})$$

$$W_E = Q_2 V_{21} + Q_3 (V_{31} + V_{32}) \rightarrow \textcircled{1}$$

If the charges were positioned in reverse order,

$$W_E = W_3 + W_2 + W_1$$

$$W_E = 0 + Q_3 V_{32} + Q_1 (V_{12} + V_{13})$$

$$W_E = Q_2 V_{23} + Q_1 (V_{12} + V_{13}) \rightarrow (2)$$

Adding (1) & (2)

$$W_E + W_E = Q_2 V_{21} + Q_3 (V_{31} + V_{32}) + Q_2 V_{23} + Q_1 (V_{12} + V_{13})$$

$$2W_E = Q_1 (V_{12} + V_{13}) + Q_2 (V_{21} + V_{23}) + Q_3 (V_{31} + V_{32})$$

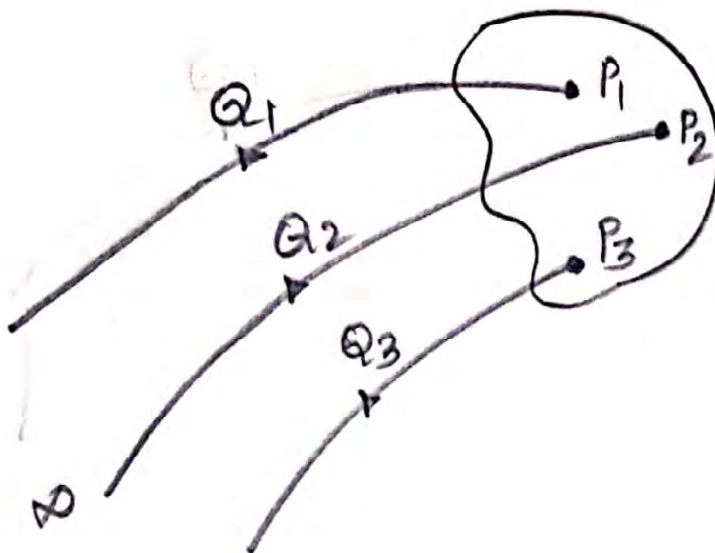
$$2W_E = Q_1 V_1 + Q_2 V_2 + Q_3 V_3$$

$$W_E = \frac{1}{2} [Q_1 V_1 + Q_2 V_2 + Q_3 V_3]$$

$$W_E = \frac{1}{2} \sum_{k=1}^3 Q_k V_k$$

If there are n point charges,

$$W_E = \frac{1}{2} \sum_{k=1}^n Q_k V_k \quad (\text{in joules})$$



Instead of point charges, the region has a continuous charge distribution, the summation becomes integration i.e.,

$$W_E = \frac{1}{2} \int_L \rho_L V dl \rightarrow \text{line charge}$$

$$W_E = \frac{1}{2} \int_S \rho_S V ds \rightarrow \text{surface charge}$$

$$W_E = \frac{1}{2} \int_V \rho_V V dv \rightarrow \text{volume charge}$$

since $\rho_V = \nabla \cdot D$ can be further developed as

$$W_E = \frac{1}{2} \int_V (\nabla \cdot D) V dv \rightarrow (1)$$

But for any vector A and scalar V , the identity

$$\nabla \cdot V\bar{A} = A \cdot \nabla V + V(\nabla \cdot A)$$

(DA)

$$(\nabla \cdot \bar{A})V = \nabla \cdot V\bar{A} - \bar{A} \cdot \nabla V$$

$$V(\nabla \cdot \bar{D}) = \nabla \cdot V\bar{D} - \bar{D} \cdot \nabla V \rightarrow (2)$$

substitute (2) in (1)

$$W_E = \frac{1}{2} \int_V (\nabla \cdot V\bar{D} - \bar{D} \cdot \nabla V) dv$$

By applying divergence theorem to first term of above eqⁿ, we get

$$W_E = \frac{1}{2} \oint_S (\nabla \bar{D}) \cdot d\bar{S} - \frac{1}{2} \int_V (\bar{D} \cdot \nabla V) dV$$

We know that V varies as $1/r$ and D as $1/r^2$ for point charges; V varies as $1/r^2$ and D as $1/r^3$ for dipoles. Hence, ∇D in 1st term on right hand side of above eqⁿ must vary atleast as $1/r^3$ while dS varies as r^2 .

$\therefore$ The 1st integral in eqⁿ must tend to zero as the surface S becomes large.

$$\therefore W_E = -\frac{1}{2} \int_V (\bar{D} \cdot \nabla V) dV$$

$$W_E = \frac{1}{2} \int_V (\bar{D} \cdot \bar{E}) dV$$

since $\bar{E} = -\nabla V$ and $\bar{D} = \epsilon_0 \bar{E}$

$$W_E = \frac{1}{2} \int_V (\bar{D} \cdot \bar{E}) dV$$

$$W_E = \frac{1}{2} \int_V \epsilon_0 \bar{E}^2 dV$$

From this, we can define electrostatic energy density W_E (in J/m³) as

$$W_E = \frac{dW_E}{dV}$$

$$W_E = \frac{1}{2} \mathbf{D} \cdot \mathbf{E}$$

$$W_E = \frac{1}{2} \epsilon_0 \mathbf{E}^2$$

$$W_E = \frac{\mathbf{D}^2}{2\epsilon_0}$$

It can be written as

$$W_E = \int_V W_E dV$$

Problem:-

- * The point charges $-1nC$, $4nC$, $3nC$ are located at $(0,0,0)$, $(0,0,1)$ and $(1,0,0)$ respectively. Find the energy in the system.

Sol:- $W = W_1 + W_2 + W_3$

$$= 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32})$$
$$= Q_2 \left[\frac{Q_1}{4\pi\epsilon_0 |(0,0,1) - (0,0,0)|} \right] +$$

$$\begin{aligned}
& \frac{Q_3}{4\pi\epsilon_0} \left[\frac{Q_1}{|(1,0,0)-(0,0,0)|} + \frac{Q_2}{|(1,0,0)-(0,0,1)|} \right] \\
&= \frac{1}{4\pi\epsilon_0} \left(Q_1Q_2 + Q_1Q_3 + \frac{Q_2Q_3}{\sqrt{2}} \right) \\
&= \frac{1}{4\pi \left(\frac{10^{-9}}{36\pi} \right)} \left(-4 - 3 + \frac{12}{\sqrt{2}} \right) \times 10^{-18} \\
&= 9 \left(\frac{12}{\sqrt{2}} - 7 \right) \text{ nJ} \\
&= 13.37 \text{ nJ}
\end{aligned}$$

convection and conduction currents:-

convection current:-

The current (in amperes) through a given area is the electric charge passing through the area per unit time.

$$I = \frac{dQ}{dt}$$

Thus in a current of one ampere, charge is being transferred at a rate of one coulomb per second.

If the current ΔI flows through a planar surface ΔS , the current density ($\vec{J}$) is

$$\vec{J} = \frac{\Delta I}{\Delta S}$$

$$\Delta I = J \Delta S$$

The total current flowing through surface S is

$$\bar{I} = \int_S J \cdot d\mathbf{s}$$

If there is a flow of charge of density ρ_v at velocity $\mathbf{u} = u_y \mathbf{a}_y$.

The current through the filament is

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \Delta S \frac{\Delta y}{\Delta t}$$

$$\Delta I = \rho_v \Delta S u_y$$

The current density at a given point is the current through a unit normal area at that point.

The y -directed current density J_y is given by.

$$J_y = \frac{\Delta I}{\Delta S} = \rho_v u_y$$

$$\boxed{\bar{J} = \rho_v \mathbf{u}}$$

The current I is the convection current and J is the convection current density in ampere per square meter (A/m^2)

conduction current:-

conduction current requires a conductor. When an electric field E is applied, the force on an electron with charge $-e$ is

$$\boxed{\vec{F} = -e\vec{E}}$$

If an electron with mass m is moving in an electric field E with an average drift velocity u , according to newton's law, the average change in momentum of free electrons must match the applied force.

$$\frac{mu}{\tau} = -e\vec{E}$$

$$\boxed{u = -\frac{e\tau}{m}\vec{E}}$$

where τ is the average time interval between collisions.

If there are n electrons per unit volume, the electronic charge density is given by

$$\boxed{\rho_V = -ne}$$

The conduction current density is

$$\vec{J} = \rho_V u$$

$$\boxed{\vec{J} = \frac{ne^2\tau}{m}\vec{E}}$$

where $\sigma = \frac{ne^2\tau}{m}$

$$\boxed{\vec{J} = \sigma \vec{E}}$$

problem:

- * A wire of diameter 1mm and conductivity $5 \times 10^7 \text{ S/m}$ has 10^{29} free electrons/ m^3 when an electric field of 10 V/m is applied. Determine
- The charge density of free electrons
 - The current density
 - The current in the wire
 - The drift velocity of the electrons. Take the electronic charge as $e = -1.6 \times 10^{-19} \text{ C}$.

sol:- In this problem, convection and conduction currents are same.

(a) $\rho_v = ne$

$$\rho_v = (10^{29}) (-1.6 \times 10^{-19})$$

$$\rho_v = -1.6 \times 10^{10} \text{ C/m}^3$$

(b) $\vec{J} = \sigma \vec{E}$

$$\vec{J} = (5 \times 10^7) (10 \times 10^{-3})$$

$$\vec{J} = 500 \text{ KA/m}^2$$

(c) $I = \vec{J} S$

$$I = (5 \times 10^5) \left(\frac{\pi d^2}{4} \right)$$

$$I = \frac{5\pi}{4} \times 10^{-6} \times 10^5$$

$$I = 0.393 \text{ A}$$

(d) since $J = \rho_v u$

$$u = \frac{\bar{J}}{\rho_v}$$

$$u = \frac{5 \times 10^5}{1.6 \times 10^{10}}$$

$$u = 3.125 \times 10^{-5} \text{ m/s.}$$

Dielectric constant:-

we know that

$$\boxed{\bar{D} = \epsilon \bar{E}}$$

where

$$\boxed{\epsilon = \epsilon_0 \epsilon_r}$$

The dielectric constant (relative permittivity) ϵ_r is the ratio of permittivity of the dielectric to that of free space.

$$\boxed{\epsilon_r = \frac{\epsilon}{\epsilon_0}}$$

where

ϵ = permittivity of the dielectric

ϵ_0 = permittivity of free space = $10^{-9} \frac{\text{F/m}}{36\pi}$

ϵ_r = Dielectric constant
(or)

Relative Permittivity

LINEAR, ISOTROPIC AND HOMOGENEOUS DIELECTRICS:-

Linear:-

A material is said to be linear if D varies linearly with E and non-linear otherwise

Homogeneous:-

Materials for which ϵ (or) σ does not vary in the region being considered and is therefore the same at all points (i.e., independent of x, y, z) are said to be homogeneous

- They are said to be non-homogeneous when ϵ is dependent of the space coordinates

ex:- Atmosphere is an inhomogeneous medium, its permittivity varies with altitude.

Isotropic:-

Materials for which D and E are in the same direction are said to be isotropic

That is, isotropic dielectrics are those which have same properties in all directions.

Anisotropic:-

Materials D , E and p are not parallel; ϵ or χ_e has nine components that are collectively referred to as a tensor

ex:-

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

crystalline materials and magnetized plasma are anisotropic

continuity equation:-

From the principle of charge conservation, the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the surface of the volume. Thus current I_{out} coming out of the closed surface is

$$I_{out} = \oint \vec{J} \cdot d\vec{s} = -\frac{dQ_{in}}{dt} \rightarrow (1)$$

Q_{in} is the total charge enclosed by closed surface.

By divergence theorem,

$$\oint_S \vec{J} \cdot d\vec{s} = \int_V \nabla \cdot \vec{J} dV \rightarrow (2)$$

But

$$-\frac{dQ_{in}}{dt} = -\frac{d}{dt} \int_V \rho_V dV = -\int_V \frac{\partial \rho_V}{\partial t} dV \rightarrow (3)$$

substituting (2) & (3) in (1)

$$\int_V \nabla \cdot \vec{J} dV = -\int_V \frac{\partial \rho_V}{\partial t} dV$$

$$\boxed{\nabla \cdot \vec{J} = -\frac{\partial \rho_V}{\partial t}}$$

which is called continuity current equation

Relaxation time:-

From conduction current,

$$\vec{J} = \sigma \vec{E}$$

From Gauss law,

$$\nabla \cdot \vec{D} = \rho_V$$

$$[D = \epsilon E]$$

$$\nabla \cdot \epsilon \vec{E} = \rho_V$$

$$\nabla \cdot \vec{E} = \frac{\rho_V}{\epsilon}$$

multiply with σ on both sides

$$\nabla \cdot \sigma \vec{E} = \sigma \frac{\rho_V}{\epsilon}$$

$$\nabla \cdot \vec{J} = \sigma \frac{\rho_V}{\epsilon}$$

$$-\frac{\partial \rho_V}{\partial t} = \frac{\sigma \rho_V}{\epsilon}$$

$$\frac{\partial \rho_V}{\partial t} = -\frac{\sigma}{\epsilon} \rho_V$$

$$\frac{\partial \rho_V}{\rho_V} = -\frac{\sigma}{\epsilon} dt$$

Applying integration on both sides

$$\int \frac{\partial \rho_V}{\rho_V} = -\frac{\sigma}{\epsilon} \int dt$$

$$\ln P_V = -\frac{\sigma}{\epsilon} t + \ln P_{V_0}$$

$$\ln P_V - \ln P_{V_0} = -\frac{\sigma}{\epsilon} t$$

$$\ln \left[\frac{P_V}{P_{V_0}} \right] = -\frac{\sigma}{\epsilon} t$$

$$\frac{P_V}{P_{V_0}} = e^{-\frac{\sigma}{\epsilon} t}$$

$$P_V = P_{V_0} e^{-\frac{\sigma}{\epsilon} t}$$

$$P_V = P_{V_0} e^{-\frac{t}{T_r}}$$

where $T_r = \frac{\epsilon}{\sigma}$

T_r = Time constant in seconds.

Relaxation time is the time it takes a charge placed in the interior of a material to drop to e^{-1} (36.8%) of its initial value.

Poisson's equation:-

Poisson's equation is easily derived from Gauss law.

$$\boxed{\nabla \cdot \vec{D} = \rho_V}$$

$$[\vec{D} = \epsilon \vec{E}]$$

$$\nabla \cdot \epsilon \vec{E} = \rho_V$$

$$\nabla \cdot \vec{E} = \frac{\rho_V}{\epsilon}$$

$$[\because \vec{E} = -\nabla V]$$

$$\nabla (-\nabla V) = \frac{\rho_V}{\epsilon}$$

$$-\nabla^2 V = \frac{\rho_V}{\epsilon}$$

$$\boxed{\nabla^2 V = -\frac{\rho_V}{\epsilon}}$$

This is called Poisson's equation

Now $\rho_V = 0$, Poisson's eqⁿ for an homogeneous medium becomes Laplacian.

$$\boxed{\nabla^2 V = 0}$$

$$\boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0}$$

capacitance:-

We define the capacitance C of the capacitor as the ratio of magnitude of the charge on one of the plates to the potential difference between them.

$$C = \frac{Q}{V}$$

$$Q = \int \rho_v dv$$

$$Q = \oint \nabla \cdot \vec{D} dv$$

$$Q = \int \vec{D} \cdot d\vec{s}$$

$$Q = \int \epsilon \vec{E} \cdot d\vec{s}$$

$$Q = \epsilon \int_S \vec{E} \cdot d\vec{s} \rightarrow (1)$$

$$V = \int_L \vec{E} \cdot d\vec{l} \rightarrow (2)$$

$$\therefore C = \frac{Q}{V}$$

$$C = \frac{\epsilon \oint_S \vec{E} \cdot d\vec{s}}{\int_L \vec{E} \cdot d\vec{l}}$$

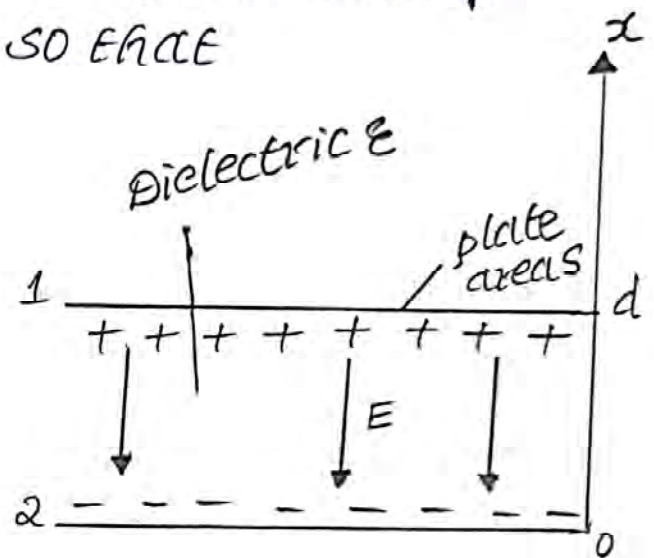
Parallel-plate capacitor:-

consider the parallel-plate capacitor. suppose that each of the plates has an area s and they are separated by a distance d . we assume that plates 1 and 2 respectively, carry charges $+Q$ and $-Q$ uniformly distributed on them so that

$$\boxed{\rho_s = \frac{Q}{s}}$$

$$\vec{E} = \frac{\rho_s}{\epsilon} (-a_x)$$

$$\boxed{\vec{E} = -\frac{Q}{s\epsilon} a_x}$$



Hence

$$V = - \int_1^2 \vec{E} \cdot d\vec{l}$$

$$V = - \int_0^d \left[-\frac{Q}{s\epsilon} a_x \right] \cdot dx a_x$$

$$\boxed{V = \frac{Qd}{s\epsilon}}$$

Thus for parallel plate capacitor,

$$C = \frac{Q}{V} = \frac{\epsilon S}{d}$$

coaxial capacitor:-

A coaxial capacitor is essentially a coaxial cable or coaxial cylindrical capacitor.

consider length L of two coaxial conductors of inner radius a and outer radius b ($b > a$). we assume that conductors 1 and 2 respectively carry $+Q$ and $-Q$ uniformly distributed on them. By applying gauss's law to an arbitrary gaussian cylindrical surface of radius r , we obtain

$$Q = \int \rho_v dv$$

$$Q = \int \nabla \cdot \vec{D} dv$$

$$Q = \int \vec{D} \cdot d\vec{s}$$

$$Q = \int \epsilon \vec{E} \cdot d\vec{s}$$

$$Q = \epsilon \int \vec{E} \cdot d\vec{s}$$

$$Q = \epsilon \bar{E}_p 2\pi r L$$

$$\bar{E}_p = \frac{Q}{\epsilon 2\pi r L}$$

$$(\int ds = 2\pi r L)$$

$$\bar{E} = \frac{Q}{\epsilon 2\pi r L} a_r$$

$$V = - \int_b^a \bar{E} \cdot d\mathbf{C}$$

$$dC = a_r dr$$

$$V = - \int_b^a \frac{Q}{\epsilon 2\pi r L} a_r dC$$

$$V = - \int_b^a \frac{Q}{\epsilon 2\pi r L} a_r \cdot a_r dr$$

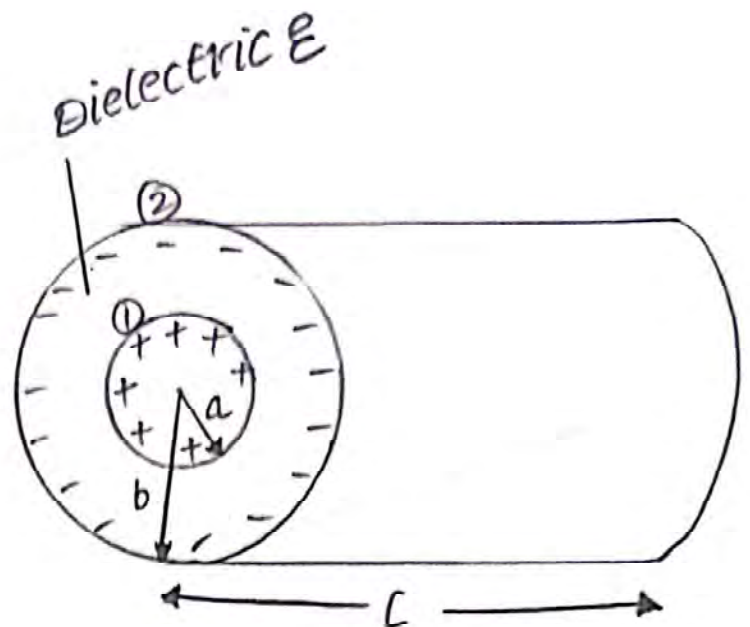
$$(a_r \cdot a_r = 1)$$

$$V = - \int_b^a \frac{Q}{\epsilon 2\pi r L} dr$$

$$V = - \frac{Q}{\epsilon 2\pi L} \int_b^a \left(\frac{1}{r} \right) dr$$

$$V = \frac{-Q}{\epsilon 2\pi L} [\ln(r)]_b^a$$

$$V = \frac{-Q}{\epsilon 2\pi L} [\ln(a) - \ln(b)]$$



$$V = \frac{Q}{\epsilon_0 2\pi L} [\ln(b) - \ln(a)]$$

$$V = \frac{Q}{\epsilon_0 2\pi L} \ln\left(\frac{b}{a}\right)$$

$$C = \frac{Q}{V}$$

$$C = \frac{Q}{\frac{Q}{\epsilon_0 2\pi L} \ln\left(\frac{b}{a}\right)}$$

capacitance of coaxial cylinder is

$$C = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)}$$

spherical capacitor:-

A spherical capacitor is the case of two concentric spherical conductors. Consider the inner sphere of radius a and outer sphere of radius b ($b > a$) separated by a dielectric medium with permittivity ϵ_0 . We assume $+Q$ and $-Q$ on the inner and outer spheres. By applying Gauss's law to an arbitrary gaussian spherical surface of radius r ($a < r < b$) we have.

$$Q = \epsilon \int E \cdot dS$$

$$Q = \epsilon E_r \oint dS$$

$$Q = \epsilon E_r 4\pi r^2$$

$$E_r = \frac{Q}{\epsilon 4\pi r^2}$$

$$E = \frac{Q}{\epsilon 4\pi r^2} \cdot a_r$$

$$V = - \int_b^a \vec{E} \cdot d\vec{l} = - \int_b^a \frac{Q}{\epsilon 4\pi r^2} a_r \cdot a_r \cdot dr$$

$$V = - \int_b^a \frac{Q}{\epsilon 4\pi r^2} dr = - \frac{Q}{\epsilon 4\pi} \int_b^a \left(\frac{1}{r^2} \right) dr$$

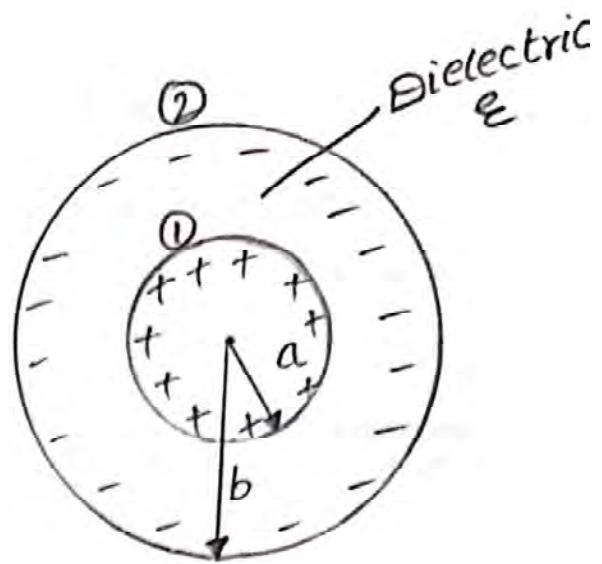
$$V = - \frac{Q}{\epsilon 4\pi} \left[-\frac{1}{r} \right]_b^a = \frac{Q}{\epsilon 4\pi} \left[\frac{1}{a} - \frac{1}{b} \right]$$

$$V = \frac{Q}{4\pi\epsilon} \left[\frac{1}{a} - \frac{1}{b} \right]$$

$$C = \frac{Q}{V}$$

$$C = \frac{Q}{\frac{Q}{4\pi\epsilon} \left[\frac{1}{a} - \frac{1}{b} \right]} = \frac{4\pi\epsilon}{\left(\frac{1}{a} - \frac{1}{b} \right)}$$

$$\boxed{C = \frac{4\pi\epsilon}{\left(\frac{1}{a} - \frac{1}{b} \right)}}$$



- : Electromagnetic waves
- And

Transmission lines :-

unit-2
u n i t

- : Magnetostatics :-
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Topics of magnetostatics:-

- 1) Introduction
- 2) Biot - Savart's Law
- 3) Ampere's circuit law
- 4) Applications of Ampere's law
- 5) magnetic flux density
- 6) Maxwell's equations for electrostatics and magnetostatics fields.
- 7) magnetic scalar and vector potentials
- 8) Forces due to Magnetic fields.
- 9) Inductances
- 10) Magnetic Energy
- 11) Faraday's law
- 12) Transformer Emf
- 13) Inconsistency of Ampere's law
- 14) Maxwell's eqns in differential form
- 15) conditions at a boundary surface.

Introduction:-

* A magnetostatic field is produced by a constant current flow. (or direct current).

* If the charges are moving with constant velocity, a static magnetic (or Magnetostatics) field is produced.

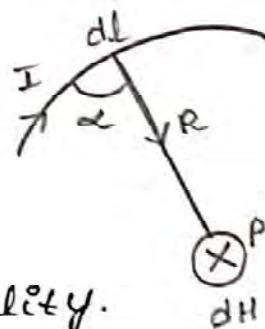
* Biot-Savart's law:-

Biot-Savart's law states that the differential magnetic field intensity dH produced at a point P , as shown in fig, by the differential current element Idl is proportional to the product Idl and the sine of the angle α between the element and the line joining P to the element and is inversely proportional to the square of the distance R between P and the element.

That is,

$$dH \propto \frac{Idl \sin \alpha}{R^2} \quad \text{--- (1)}$$

$$dH = \frac{k Idl \sin \alpha}{R^2} \quad \text{--- (2)}$$



Where k is the constant of proportionality.

In SI units, $k = \frac{1}{4\pi}$

Substituting ' k ' in (2), we get

$$dH = \frac{Idl \sin \alpha}{4\pi R^2} \quad \text{--- (3)}$$

From the definition of cross product, eq (3) in the vector form is given as

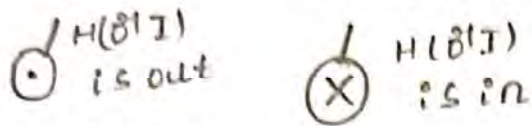
$$dH = \frac{Idl \times a_R}{4\pi R^2} = \frac{Idl \times \bar{R}}{4\pi R^3} \quad \text{--- (4)}$$

Where $a_R = \frac{\bar{R}}{|\bar{R}|} = \frac{\bar{R}}{R}$

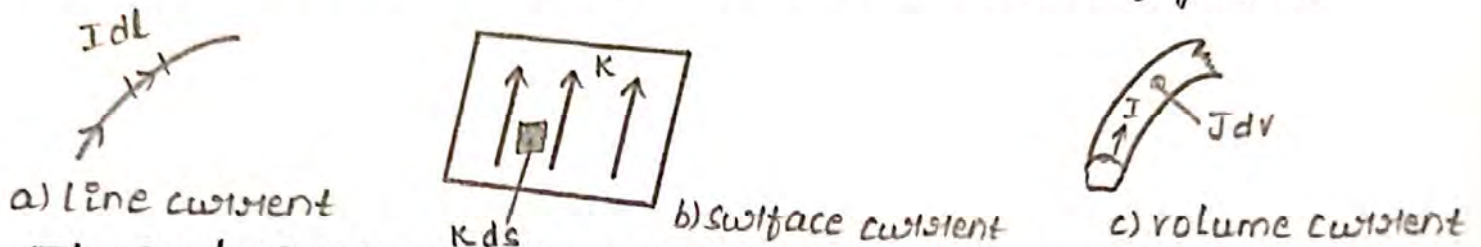
$\bar{R}$ and dl are illustrated in fig.

Thus the direction of dH can be determined by the right-hand rule with the right-hand thumb pointing in the direction of the current and the right-hand fingers encircling the wire in the direction of dH .

It is customary to represent the direction of the magnetic field intensity H by a small circle with a dot or cross sign depending on whether H is out of, or into the page.



We can have different current distributions; line current, surface current and volume current as shown in fig below.



If we define K as the surface current density in amperes per meter and J as the volume current density in amperes per meter squared, the source elements are related as

$$Idl = Kds = Jdv$$

Thus in terms of the distributed current sources the biot-savarts law becomes.

$$H = \int_L \frac{Idl \times a_R}{4\pi R^2} \quad (\text{line current})$$

$$H = \int_S \frac{Kds \times a_R}{4\pi R^2} \quad (\text{surface current})$$

$$H = \int_V \frac{Jdv \times a_R}{4\pi R^2} \quad (\text{volume current})$$

where a_R is a unit vector pointing from the differential element of current to the point of interest.

* Ampere's circuit law:-

Ampere's circuit law states that the line integral of H around a closed path is the same as the net current I_{enc} enclosed by the path.

In other words, the circulation of H equals I_{enc} , that is

$$\oint H \cdot dl = I_{enc} \quad \text{--- (1)}$$

Ampere's law is similar to Gauss law, since Ampere's law is easily applied to determine H when the current distribution is similar/symmetrical.

It should be noted that eq (1) always holds regardless of whether the current distribution is symmetrical, but we can use the eqⁿ to determine H only when a symmetrical current distribution exists.

Ampere's law is a special case of Biot-Savart's law, the former may be derived from the latter.

By applying Stokes's theorem to the left-hand side of eq (1), we get

$$I_{enc} = \oint_L H \cdot dl = \int_S (\nabla \times H) \cdot dS \quad \text{--- (2)}$$

$$\text{But } I_{enc} = \int_S J \cdot dS \quad \text{--- (3)}$$

Comparing the surface integrals in eq (2) & (3)

$$\nabla \times H = J \quad \text{--- (4)}$$

This is the third Maxwell's eqⁿ to be derived, it is essentially Ampere's law in differential form. Whereas eq (1) is the integral form.

Applications of Ampere's law:-

We apply Ampere's circuit law to determine H for some symmetrical current distributions.

We will consider an infinite line current, an infinite current sheet of, and an infinitely long coaxial transmission line.

In each case, we apply

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = I_{enc}$$

For symmetrical current distribution, $\mathbf{H}$ is either parallel or perpendicular to $d\mathbf{l}$.

When $\mathbf{H}$ is parallel to $d\mathbf{l}$, $|\mathbf{H}| = \text{constant}$

1) Infinite line current:-

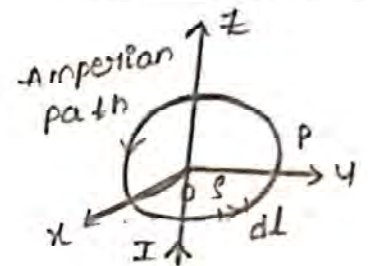
consider an infinitely long filamentary current I along the z -axis as shown in fig.

To determine $\mathbf{H}$ at an observation point P , we allow a closed path to pass through P .

This path, on which Ampere's law is to be applied, is known as an Amperian path.

We choose a concentric circle as the Amperian path, which shows that $\mathbf{H}$ is constant provided ρ is constant.

Since this path encloses the whole current I , according to Ampere's law,

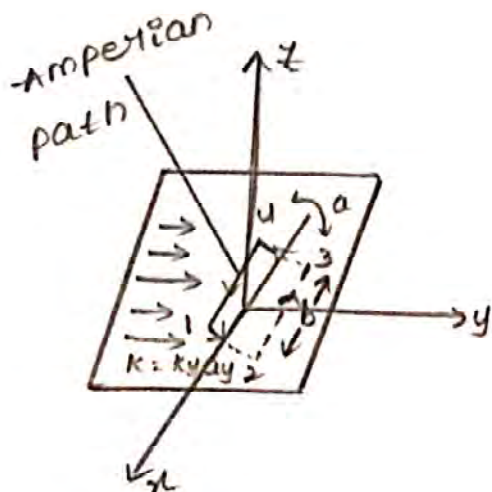


$$I = \int \mathbf{H} \cdot d\mathbf{l} = \int H \cdot \rho d\phi = H \rho \int d\phi = H \rho \cdot 2\pi\rho$$

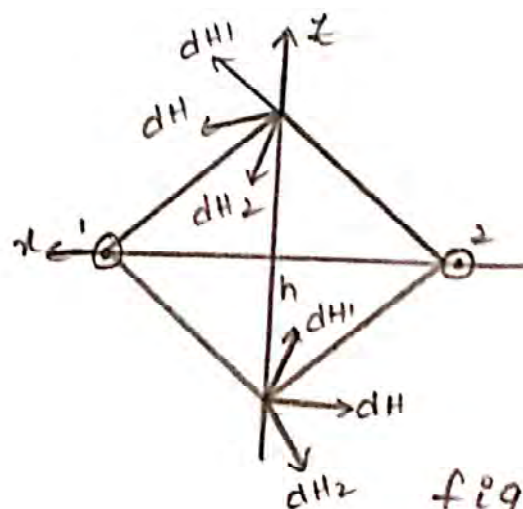
$$\Rightarrow H = \frac{I}{2\pi\rho} \hat{\phi}$$

2) Infinite sheet of current:-

consider an infinite current sheet in the $z=0$ plane. If the sheet has a uniform current density $\mathbf{K} = K_y \hat{y}$ A/m.



fig(a)



fig(b)

applying Ampere's law to the rectangular closed path 1-2-3-4-1 (Amperean path) gives

$$\oint H \cdot dl = I_{enc} = kyb \quad \text{--- (1)}$$

To evaluate the integral, we first need to have an idea of what H is like. To achieve this, we regard the infinite sheet as comprising of filaments; dH above & below the sheet due to a pair of filamentary currents.

As evident in fig, the resultant dH has only an x -component. Also, H on one side of the sheet is the negative of that on the other side.

Owing to the infinite extent of the sheet, the sheet can be regarded as consisting of such filamentary pairs so that the characteristics of H for a pair are the same for the infinite current sheets,

that is,

$$H = \begin{cases} H_0 ax & z > 0 \\ -H_0 ax & z < 0 \end{cases} \quad \text{--- (2)}$$

Evaluating the line integral of H in eq (2) along the closed path in fig gives

$$\begin{aligned} \oint H \cdot dl &= \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right) H \cdot dl \\ &= 0(-a) + (-H_0)(-b) + 0(a) + H_0(b) \end{aligned}$$

$$\Rightarrow \oint H \cdot dl = 2H_0b \quad \text{--- (3)}$$

From eq (1) & (3), we obtain $H_0 = \frac{1}{2} ky$.

Substituting H_0 in eq (2) gives

$$H = \begin{cases} \frac{1}{2} ky ax & z > 0 \\ -\frac{1}{2} ky ax & z < 0 \end{cases} \quad \text{--- (4)}$$

In general, for an infinite sheet of current density K A/m,

$$H = \frac{1}{2} K \times a_n \quad \text{--- (5)}$$

Where a_n is a unit normal vector directed from the current sheet to the point of interest.

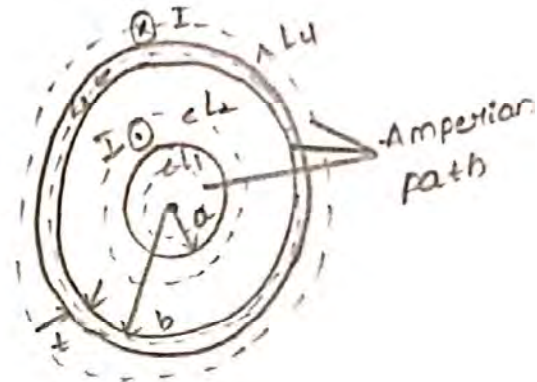
3) Infinitely long coaxial Transmission line:-

Consider an infinitely long transmission line consisting of two concentric cylinders having their axes along the z -axis. The cross section of the line, where z -axis is out of the page.

The inner conductor has radius a and carries current I , while the outer conductor has inner radius b and thickness t and carries return current $-I$.

We want to determine H everywhere, assuming that current is uniformly in both conductors.

$$H = \begin{cases} \frac{I\rho}{2\pi a^2} a\phi & 0 \leq \rho \leq a \\ \frac{I}{2\pi\rho} a\phi & a \leq \rho \leq b \\ \frac{I}{2\pi\rho} \left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt} \right] a\phi & b \leq \rho \leq b+t \\ 0 & \rho \geq b+t \end{cases}$$



* Magnetic Flux Density:-

The magnetic flux density B is similar to the electric flux density. As D to E in free space, the magnetic flux density B is related to the magnetic field intensity H according to

$$B = \mu_0 H \quad \text{--- (1)}$$

Where μ_0 is a constant known as the permeability of free space.

The constant is in henry's per meter (H/m) and has the value of

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m.}$$

The magnetic flux through a surface S is given by

$$\psi = \int_S B \cdot dS \quad \text{--- (2)}$$

Where magnetic flux ψ is in webers (Wb) and magnetic flux density is in webers per square meters (Wb/m²) or Tesla (T).

The total flux through a closed surface in a magnetic field must be zero, that is

$$\oint \mathbf{B} \cdot d\mathbf{s} = 0 \quad - (3)$$

This equation is referred to as the law of conservation of magnetic flux or Gauss law for magnetostatic fields, just as $\oint \mathbf{D} \cdot d\mathbf{s} = Q$ is Gauss law for electrostatic fields.

Although the magnetostatic field is not conservative, magnetic flux is conserved.

By applying the divergence theorem to eq (3), we get

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = \int_V \nabla \cdot \mathbf{B} dv = 0$$

$$\Rightarrow \nabla \cdot \mathbf{B} = 0 \quad - (4) \quad [\text{Maxwell's eq}]$$

Eq (3) & (4) shows that the magnetostatic fields have no sources or sinks.

Eq (4) suggests that magnetic field lines are always continuous.

* Maxwell's eqns for Electrostatic and Magnetostatic fields:-

Differential form	Integral form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho_v dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$	Nonexistence of magnetic monopole.
$\nabla \times \mathbf{E} = 0$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = 0$	Conservative nature of Electrostatic field.
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{s}$	Ampere's law

* Magnetic Scalar and vector potentials:-

The magnetic potential could be scalar V_m or vector A .

To define V_m and A involves recalling 2 important identities:

$$\nabla \times (\nabla V) = 0 \quad - (1)$$

$$\nabla \cdot (\nabla \times A) = 0 \quad - (2)$$

Which must always hold for any scalar field V and vector field A . Just as $E = -\nabla V$, we define the "magnetic scalar potential" V_m as related to H according to

$$H = -\nabla V_m \quad \text{if } J=0 \quad - (3)$$

From Ampere's law,

$$\nabla \times H = J$$

$$J = \nabla \times H \quad - (4)$$

Substituting eq (3) in (4), we get

$$J = \nabla \times (-\nabla V_m) = 0 \quad - (5)$$

Since V_m must satisfy the condition in eq (1) thus the magnetic scalar potential V_m is only defined in a region where $J=0$ as in eq (3).

We should also note that V_m satisfies Laplace's eqn just as V does for electrostatic fields,

$$\text{hence, } \nabla^2 V_m = 0 \quad (J=0) \quad - (6)$$

We know that for a magnetostatic field,

$$\nabla \cdot B = 0 \quad - (7)$$

To satisfy eq (7) & (2) simultaneously, we can define the vector magnetic potential (A) (in wb/m) such that

$$B = \nabla \times A \quad - (8)$$

Just as we defined

$$V = \int \frac{dq}{4\pi\epsilon_0 r}$$

We can define

$$A = \int_L \frac{\mu_0 I dl}{4\pi R} \quad (\text{Line current}) - (9)$$

$$A = \int_S \frac{\mu_0 K ds}{4\pi R} \quad (\text{Surface current}) - (10)$$

$$A = \int_V \frac{\mu_0 J dv}{4\pi R} \quad (\text{Volume current}) - (11)$$

* Forces due to Magnetic Fields:-

There are at least 3 ways in which force due to magnetic fields can be experienced.

The force can be

- 1) due to a moving charged particle in a B field.
- 2) On a current element in an external B field, or
- 3) between two current elements.

1) Force on a charged particle:-

The electric force F_e on a stationary or moving electric charge Q in an electric field is given by Coulomb's experimental law and is related to the electric field intensity E as

$$F_e = QE - (1)$$

This shows that if Q is positive, F_e and E have the same direction.

A magnetic field can exert force only on a moving charge.

From experiments, it is found that the magnetic force F_m experienced by a charge Q moving with a velocity u in a magnetic field B is

$$F_m = Qu \times B - (2)$$

This clearly shows that F_m is perpendicular to both u and B .

Comparison between the electric force F_e and the magnetic

Force F_m :- F_e is independent of the velocity of the charge

and can perform work on the charge and change its kinetic energy.

- unlike F_e , F_m depends on the charge velocity and is normal to it.
- The magnitude of F_m is generally small in comparison to F_e except at high velocities.

For a moving charge q in the presence of both electric and magnetic fields, total force on the charge is given by

$$F = F_e + F_m$$

$$F = q(E + u \times B) \quad \text{--- (3)}$$

This is known as the Lorentz force equation.

It relates mechanical force to electrical force.

If the mass of the charged particle moving in E and B fields is m , by Newton's Second Law of motion.

$$F = m \frac{du}{dt} = q(E + u \times B)$$

The solution to this eqn is important in determining the motion of charged particles in E and B fields.

Table :- Force on a charged particle

State of particle	E field	B field	Combined E and B fields
Stationary	qE	—	qE
moving	qE	$qu \times B$	$q(E + u \times B)$

2) Force on a current Element :-

To determine the force on a current element Idl of a current carrying conductor due to the magnetic field B , we modify eqn as $F_m = q u \times B$ using the fact that for convection current

$$J = \int_V u \quad \text{--- (1)}$$

We know the relation between current elements :-

$$Idl = k ds = J dv \quad \text{--- (2)}$$

Combining eqn (1) & (2), we get

$$Idl = \int v u dv = dqu$$

Alternatively,

$$Idl = \frac{dq}{dt} dl = dq \frac{dl}{dt} = dqu$$

Hence

$$Idl = dqu - (3)$$

This shows that an elemental charge dq moving with velocity u is equivalent to a conduction current element Idl .

Thus the force on a current element Idl in a magnetic field B is found from $F_m = qu \times B$ by merely replacing qu by Idl , that is, $dF = Idl \times B - (4)$

If the current I is through a closed path L of circuit, the force on the circuit is given by

$$F = \oint_L Idl \times B - (5)$$

For surface and volume currents, eq (4) becomes

$$dF = k dS \times B$$

$$dF = J dV \times B$$

eq (5) becomes,

$$F = \int_S k dS \times B$$

$$F = \int_V J dV \times B$$

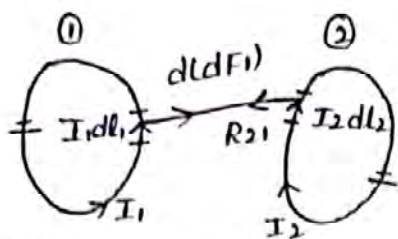
From
eq (4)

The magnetic field ' B ' is defined as the force per unit current element.

3) Force between Two Current Elements:-

Let us consider the force between two elements $I_1 dl_1$ and $I_2 dl_2$. According to Biot-Savart's law, both current elements produce magnetic fields.

So we may find the force $d(F_1)$ on element $I_1 dl_1$ due to the field dB_2 produced by element $I_2 dl_2$ as shown in fig.



From eqn $dF = Idl \times B$,

$$d(dF_1) = I_1 dl_1 \times dB_2 \quad \text{--- (1)}$$

But from biot-savart's law,

$$dB_2 = \frac{\mu_0 I_2 dl_2 \times a_{R21}}{4\pi R_{21}^2} \quad \text{--- (2)}$$

Hence,

$$d(dF_1) = \frac{\mu_0 I_1 dl_1 \times (I_2 dl_2 \times a_{R21})}{4\pi R_{21}^2} \quad \text{--- (3)}$$

This equation is essentially the law of force between two current elements and is analogous to coulombs law, which expresses the force between two stationary charges.

From eq (3), we obtain the total force F_1 on current loop 1 due to current loop 2 shown in fig as

$$F_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{dl_1 \times (dl_2 \times a_{R21})}{R_{21}^2} \quad \text{--- (4)}$$

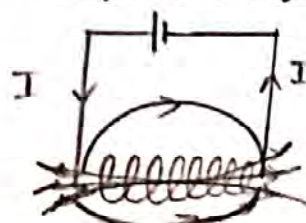
The force F_2 on loop 2 due to the magnetic field B_1 from loop 1 is obtained from eq (4) by interchanging subscripts 1 and 2. It can be shown that $F_2 = -F_1$, thus F_1 and F_2 obey Newton's third law that action and reaction are equal and opposite.

* Inductances :-

A circuit (or closed conducting path) carrying current I produces a magnetic field B that ~~become~~ causes a flux $\psi = \int B \cdot ds$ to pass through each turn of the circuit as shown in fig.

If the circuit has N identical turns, we define the flux linkage λ as

$$\lambda = N\psi \quad \text{--- (1)}$$



Also, if the medium surrounding the circuit is linear, the flux linkage λ is proportional to the current I producing it, that is

$$\lambda \propto I$$
$$\lambda = LI - (2)$$

Where L is a constant of proportionality called the inductance of the circuit.

- The inductance L is a property of the physical arrangement of the circuit.
- A circuit or part of a circuit that has inductance is called an inductor.

From eq (1) & (2), we may define inductance L of an inductor as the ratio of the magnetic flux linkage λ to the current I through the inductor,

$$L = \frac{\lambda}{I} = \frac{N\psi}{I} - (3)$$

- The unit of inductance is the henry (H) which is the same as the volt second per ampere.
- The inductance defined by eq (3) is commonly referred to as "self-inductance", since the linkages are produced by the inductor itself.

Like capacitances, inductance may be regarded as a measure of how much magnetic energy is stored in an inductor.

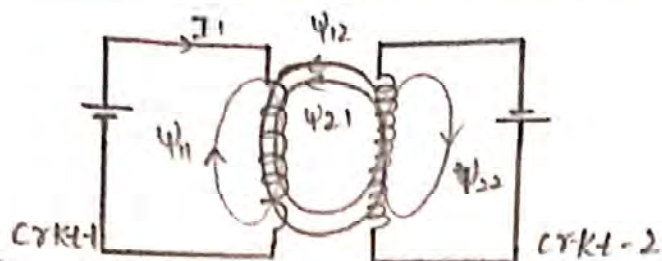
The magnetic energy stored in an inductor is expressed in circuit theory as

$$W_m = \frac{1}{2} LI^2 - (4)$$

$$L = \frac{2W_m}{I^2} - (5)$$

Thus the self-inductance of a circuit may be defined or calculated from energy considerations.

If instead of having a single circuit, we have two circuits carrying current I_1 and I_2 as shown in fig below, a magnetic interaction exists between the circuits.



Four component fluxes ψ_{11} , ψ_{12} , ψ_{21} , and ψ_{22} are produced.

The flux ψ_{12} , for example, is the flux passing through circuit 1 due to current I_2 in circuit 2.

If B_2 is the field due to I_2 and S_1 is the area of circuit 1, then

$$\psi_{12} = \int_{S_1} B_2 \cdot dS \quad \text{--- (6)}$$

The "mutual inductance" is defined as the ratio of the flux linkage

$\lambda_{12} = N_1 \psi_{12}$ on circuit 1 to current I_2 ;

that is,

$$M_{12} = \frac{\lambda_{12}}{I_2} = \frac{N_1 \psi_{12}}{I_2} \quad \text{--- (7)}$$

It can be shown by using energy concepts that if the medium surrounding the circuits is linear,

$$M_{12} = M_{21} \quad \text{--- (8)}$$

Similarly, the mutual inductance M_{21} is defined as the flux linkages of circuit 2 to current I_1 ;

that is,

$$M_{21} = \frac{\lambda_{21}}{I_1} = \frac{N_2 \psi_{21}}{I_1} \quad \text{--- (9)}$$

We define the self-inductance of circuits 1 and 2, resp, as

$$L_1 = \frac{\lambda_{11}}{I_1} = \frac{N_1 \psi_1}{I_1}$$

$$L_2 = \frac{\lambda_{22}}{I_2} = \frac{N_2 \psi_2}{I_2}$$

$$\text{where } \psi_1 = \psi_{11} + \psi_{12}$$

$$\psi_2 = \psi_{21} + \psi_{22}$$

The total energy in the magnetic field is the sum of the energies due to L_1 , L_2 , and M_{12} (or M_{21}),

that is

$$W_m = W_1 + W_2 + W_{12}$$

$$= \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 \pm M_{12} I_1 I_2 \quad \text{--- (10)}$$

The +ve sign is taken if currents I_1 and I_2 flow such that the magnetic fields of the two circuit strengthen each other.

If the currents flow such that their magnetic fields oppose each other, the -ve sign is taken.

In an inductor such as a coaxial or a parallel-wire transmission line, the inductance produced by the flux internal to the conductor is called the "internal inductance" L_{in} while that produced by the flux external to it is called "external inductance" L_{ext} .

The total inductance L is

$$L = L_{in} + L_{ext} \quad - (11)$$

$$L_{ext} C = \mu \epsilon \quad - (12)$$

* Magnetic Energy:-

The magnetic energy in the field of an inductor is $w_m = \frac{1}{2} L I^2$ - (1)

The energy is stored in the magnetic field B of the inductor.

Consider a differential volume in a magnetic field as shown in fig below.

Let the volume be covered with conducting sheets at the top and bottom surfaces with current ΔI .

The inductance is $L = \frac{N\psi}{I}$

$$\Delta L = \frac{\Delta \psi}{\Delta I} = \frac{\mu H \Delta x \Delta y \Delta z}{\Delta I} \quad - (2)$$

where $\Delta I = H \Delta y$

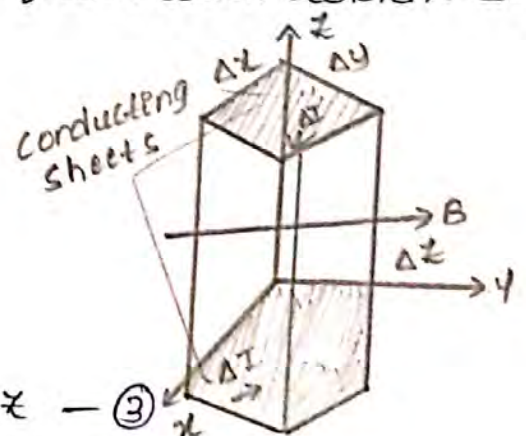
eq (2) in (1),

$$\Delta w_m = \frac{1}{2} \Delta L \Delta I^2 = \frac{1}{2} \mu H^2 \Delta x \Delta y \Delta z \quad - (3)$$

$$\Delta w_m = \frac{1}{2} \mu H^2 \Delta V$$

The magnetostatic energy density w_m (in J/m³) is defined as $w_m = \lim_{\Delta V \rightarrow 0} \frac{\Delta w_m}{\Delta V} = \frac{1}{2} \mu H^2$

$$w_m = \frac{1}{2} \mu H^2 = \frac{1}{2} B \cdot H = \frac{B^2}{2\mu} \quad - (4)$$



Thus, the energy in a magnetostatic field in a linear medium is $W_m = \int w_m dv$

$$W_m = \frac{1}{2} \int \mathbf{B} \cdot \mathbf{H} dv = \frac{1}{2} \int \mu H^2 dv \quad (5)$$

eq (5) is similar to the potential energy in an electrostatic field W_e .

$$W_e = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} dv = \frac{1}{2} \int \epsilon E^2 dv //$$

* Faraday's law:-

Oersted discovered that magnetic field is produced by changing electric current in 1820.

In 1831, 11 years after Oersted's discovery, Michael Faraday discovered that a time varying magnetic field would produce an electric current.

According to Faraday's experiments, a static magnetic field produces no current flow, but a time-varying field produces an induced voltage (called emf) in a closed circuit, which causes a flow of current.

Faraday discovered that the induced emf, V_{emf} in any closed circuit is equal to the time rate of change of magnetic flux linkage by the circuit.

This is called Faraday's law, and it can be expressed as

$$V_{emf} = - \frac{d\lambda}{dt} = - N \frac{d\psi}{dt} \quad (1)$$

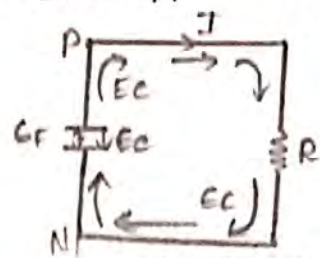
Where $\lambda = N\psi$ is the flux linkage,

N = number of turns in the circuit.

ψ = flux through each turn.

The negative sign shows that the induced voltage acts in such a way to oppose the flux producing it. This is known as Lenz's law, and it emphasizes that the direction of current flow in the circuit is such that the induced

magnetic field produced by the induced current will oppose the change in signal magnetic field.



Consider the electric circuit, where the battery is a source of emf.

The electrochemical action of the battery results in an emf-produced field E_f . Due to the accumulation of charge at the battery terminals, an electrostatic field $E_e = -\nabla V$ also exists.

The total electric field at any point is

$$E = E_f + E_e \quad (2)$$

If we integrate eq (2) over the closed circuit, then

$$\oint E \cdot d\mathbf{l} = \oint E_f \cdot d\mathbf{l} + 0 = \int_N^P E_f \cdot d\mathbf{l} \quad (3)$$

Where $\oint E_e \cdot d\mathbf{l} = 0$ because E_e is conservative.

$$V_{emf} = \int_N^P E_f \cdot d\mathbf{l} = - \int_N^P E_e \cdot d\mathbf{l} = IR \quad (4)$$

Since E_f and E_e are equal but opposite in direction within the battery.

Note:-

1) An electrostatic field E_e cannot maintain a steady current in a closed circuit since

$$\oint E_e \cdot d\mathbf{l} = 0 = IR$$

2) An emf-produced field E_f is non conservative.

3) Except in electrostatics, voltage and potential difference are usually not equivalent.

* Transformer Emf:-

The emf induced in a stationary conductor due to the change in flux linked with it, is called transformer emf.

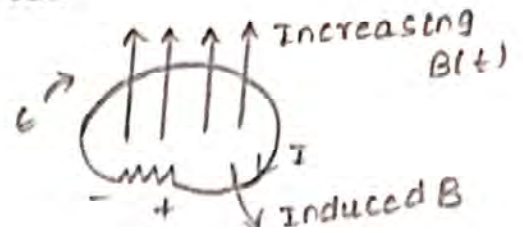
$$V_{emf} = \oint E \cdot d\mathbf{l} = - \int_S \frac{dB}{dt} \cdot d\mathbf{S}$$

The emf induced by the time-varying current in a stationary loop is often referred to as transformer emf in power analysis. Since it is due to transformer action.

By applying Stoke's theorem to the middle term in above eqn.

$$\oint_S (\nabla \times E) \cdot dS = - \int_S \frac{\partial B}{\partial t} dS.$$

$$\nabla \times E = - \frac{\partial B}{\partial t} \text{ n.}$$



Inconsistency of Ampere's Law:-

Ampere's law states that the line integral of the H around a closed path is same as the net current I_{enc} closed by the path.

$$\oint_L H \cdot dl = I_{enc} = \int_S J \cdot dS \quad (1)$$

By applying Stoke's theorem

$$\oint_L H \cdot dl = \int_S (\nabla \times H) \cdot dS \quad (2)$$

Comparing above eqns

$$\nabla \times H = J \quad (3)$$

Taking divergence

$$\nabla \cdot (\nabla \times H) = \nabla \cdot J = 0$$

But the eqn of continuity for time varying field is

$$\nabla \cdot J = - \frac{\partial \rho}{\partial t} \quad (4)$$

Thus Ampere's law is not consistent with the time varying eqn of continuity.

Maxwell suggested a modification to Ampere's law to avoid inconsistency.

According to Gauss law, the divergence of electric flux density is equal to the volume charge density.

$$\nabla \cdot D = \rho \quad (5)$$

eq (5) in (4)

$$\nabla \cdot J = - \frac{\partial}{\partial t} \nabla \cdot D$$

$$= - \nabla \cdot \frac{\partial D}{\partial t}$$

$$\nabla \cdot \mathbf{J} + \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} = 0$$

$$\nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) = 0 \quad \text{--- (6)}$$

Apply divergence theorem to eqn (6)

$$\int_V \nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) dV = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S} = 0$$

$$\Rightarrow \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S} = 0$$

$\frac{\partial \mathbf{D}}{\partial t}$ is called as the displacement current density.

Maxwell modified the ampere's law by replacing $\mathbf{J}$ by total current density.

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$$

By applying Stoke's theorem

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$$

$$\therefore \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Taking divergence

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) = 0$$

$$\Rightarrow \nabla \cdot \mathbf{J} = -\nabla \cdot \frac{\partial \mathbf{D}}{\partial t} = -\frac{\partial \rho}{\partial t}$$

It becomes the continuity eqn.

Thus the inconsistency has been removed.

* Maxwell's eqns in different final forms:-

Differential form	Integral form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho_v dV$	Gauss law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Non-existence of isolated magnetic charge
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$	Ampere's circuit law

* Conditions at a Boundary Surface:-

Maxwell's eqns in the differential form express the relationship that must exist between the four field vectors E, D, H and B at any point within a continuous medium.

However, the integral forms can always be used to determine what happens at the boundary surface between different medium.

The following statements can be made regarding the electric and magnetic fields at any surface of discontinuity:

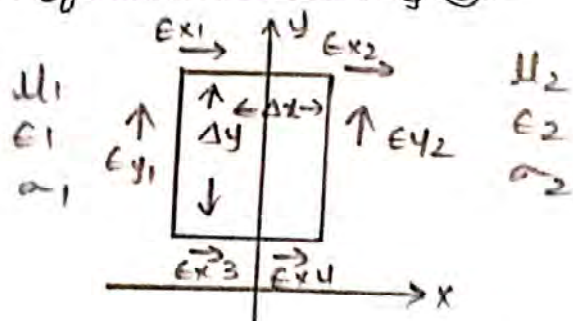
- The tangential component of E is continuous at the surface. That is it is the same just outside the surface as it is just inside the surface.
- The tangential component of H is continuous across a surface except at the surface of a perfect conductor. At the surface of a perfect conductor the tangential component of H is discontinuous by an amount equal to the surface current per unit width.
- The normal component of B is continuous at the surface of discontinuity.
- The normal component of D is continuous if there is no surface charge density. Otherwise D is discontinuous by an amount equal to the surface charge density.

The proof of these boundary conditions is obtained by a direct application of Maxwell's equation at the boundary between the medium.

Suppose the surface of discontinuity to be the plane $x=0$ as shown in fig. consider the small rectangle of width Δx and length Δy enclosing a small portion of each media ① & ②.

The integral form of the second Maxwell eqn is

$$\oint E \cdot ds = - \int \frac{dB}{dt} \cdot da \quad \text{--- (1)}$$



For the elemental rectangle of fig this becomes.

$$E_{y2} \Delta y - E_{x2} \frac{\Delta x}{2} - E_{x1} \frac{\Delta x}{2} - E_{y1} \Delta y + E_{x3} \frac{\Delta x}{2} + E_{x4} \frac{\Delta x}{2} = -\frac{\partial B_z}{\partial t} \Delta x \Delta y.$$

Where B_z is the average magnetic flux density through the rectangle $\Delta x \Delta y$.

Now consider conditions as the area of the rectangle is made to approach zero by reducing the width Δx of the rectangle, always keeping the surface of discontinuity between the sides of the rectangle.

If it is assumed to be everywhere finite, then the $\Delta x/2$ terms of the left-hand side will reduce to zero, leaving

$$E_{y2} \Delta y - E_{y1} \Delta y = 0 \quad \text{for } \Delta x = 0$$

$$\Rightarrow E_{y1} = E_{y2}$$

That is, the tangential component of E is continuous

Similarly the integral statement of first Maxwell's eqn is

$$\oint \mathbf{H} \cdot d\mathbf{s} = \int \left(\frac{\partial D}{\partial t} + J \right) da \quad \text{--- (3)}$$

which becomes

$$H_{y2} \Delta y - H_{x2} \frac{\Delta x}{2} - H_{y1} \Delta y + H_{x3} \frac{\Delta x}{2} + H_{x4} \frac{\Delta x}{2} = \left(\frac{\partial D_z}{\partial t} + J_z \right) \Delta x \Delta y \quad \text{--- (4)}$$

If the rate of change of electric displacement D and current density J are both considered to be finite, then as before eq (4) reduces to $H_{y2} \Delta y - H_{y1} \Delta y = 0$

$$H_{y1} = H_{y2}$$

The tangential component of H is continuous.

Perfect conductor: - A perfect conductor is one which has infinite conductivity. In such a conductor the electric field strength E is zero for any finite current density. In a good conductor a high-frequency current will flow in a thin sheet near the surface, the depth of this sheet approaching zero as the conductivity approaches infinity. This gives rise to the useful concept of a current sheet. In a current sheet a

finite current per unit width, J_s amperes per meter, flows in a sheet of small depth Δx , but with the required infinitely large current density J , such that

$$\lim_{\Delta x \rightarrow 0} J \Delta x = J_s \text{ A/m.}$$

If the current density J_x becomes infinite as Δx approaches zero the right-hand side of eq (4) will not become zero.

Let J_s ampere's per meter be the actual current per unit width flowing along the surface.

Then as $\Delta x \rightarrow 0$ the eq (4) for H becomes

$$H_{y2} \Delta y - H_{y1} \Delta y = J_s x \Delta y$$

$$H_{y1} = H_{y2} - J_s x \quad (5)$$

Now, if the electric field is zero within a perfect conductor, the magnetic field must also be zero as the second maxwell eqn shown. Then in eq (5), the H_{y2} must be zero and

$$\text{So } H_{y1} = -J_s x \quad (6)$$

eq (6) states that the current per unit width along the surface of a perfect conductor is equal in the magnetic field strength H just outside the surface.

The magnetic field and surface current will be parallel to the surface, but perpendicular to each other. In vector notation this is written as

$$J_s = \hat{n} \times H.$$

Where $\hat{n}$ is the unit vector along the outward normal to the surface.

Conditions for on the normal components of B and D :

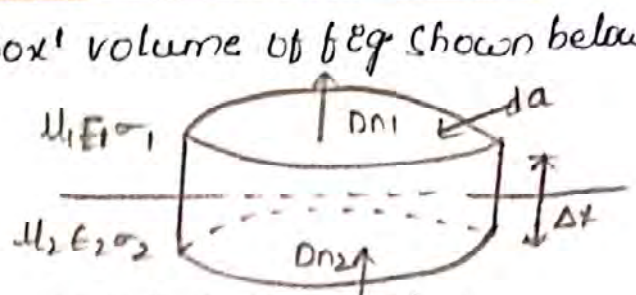
The remaining boundary conditions are concerned with the normal components of B and D .

The integral form of the third field eqn is

$$\oint_C D \cdot da = \int V \rho dv$$

When applied to the elementary "pill-box" volume of fig shown below. The above eqn becomes

$$D_{n1} da - D_{n2} da + \psi_{\text{edge}} = \int \Delta x da \quad - (7)$$



da is the area of each of the flat surfaces of the pill box, Δx is the separation and is the average charge density within the volume $\Delta x da$. ψ_{edge} is the outward electric flux through the curved-edge surface of the pill box.

As $\Delta x \rightarrow 0$ that is, as the flat surfaces of the box are squeezed together, always keeping the boundary surface between them, $\psi_{\text{edge}} \rightarrow 0$ for finite values of displacement density.

Also for finite values of displacement density, average charge density ρ , the right hand side of eq (7) approaches zero, and eq (7) reduces to

$$D_{n1} da - D_{n2} da = 0$$

$$\Rightarrow D_{n1} = D_{n2} \quad (\text{for } \Delta x = 0) \quad - (8)$$

That is, if there is no surface charge the normal component of D is continuous across the surface.

In the case of a metallic surface, the charge is considered to reside "on the surface". If this layer of surface charge has a surface charge density ρ_s coulombs per square meter, the charge density ρ of the surface layer is given by

$$\rho = \frac{\rho_s}{\Delta x} \quad \text{coulomb/cm}.$$

where Δx is thickness of the surface layer.

As Δx approaches zero, the charge density approaches infinity.

$$\lim_{\Delta x \rightarrow 0} \rho \Delta x = \rho_s$$

Then in fig shown above, if the surface charge is always kept between the two flat surface as the separation between them is decreased, the right-hand side of eq (9) approaches $\int_S da$ as Δx approaches zero.

eqn (9) then reduces to

$$D_{n1} - D_{n2} = \int_S - (10)$$

When there is a surface charge density ρ_s , the normal component of displacement density is discontinuous across the surface by the amount of the surface charge density.

For an metallic conductor the displacement density $D = \epsilon E$ within the conductor will be a very small quantity.

Then if medium (2) is a metallic conductor $D_{n2} = 0$ and eq (9) becomes

$$D_{n1} = \rho_s - (10)$$

The normal component of displacement density in the dielectric is equal to the surface charge density on the conductor.

In the case of magnetic flux density B , since there are no isolated "magnetic charges", a similar analysis leads at once to

$$B_{n1} = B_{n2}$$

The normal component of magnetic flux density is always continuous across a boundary surface.

ELECTRO MAGNETIC WAVES AND TRANSMISSION LINES

TOPICS OF UNIT-3:- EM wave characteristics - I

1. Wave Equations for conducting and Perfect Dielectric media.
2. Uniform plane waves - Definition.
3. All Relations between E & H .
4. Sinusoidal Variations.
5. Wave propagation in different mediums.
6. Polarization, Illustrative Problems.
7. Reflection and Refraction of Plane Waves; - Normal and Oblique Incidences for both perfect conductor and perfect Dielectrics.
8. Brewster Angle.
9. Critical Angle and Total Internal Reflection.
10. Surface Impedance.
11. Introduction to Poynting Theorem - Applications, Illustrative problems.

Introduction :-

In the solution of any electromagnetic problem the fundamental relations that must be satisfied, are 4 field equations:

$$\rightarrow \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\rightarrow \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\rightarrow \nabla \cdot \vec{D} = \rho$$

$$\rightarrow \nabla \cdot \vec{B} = 0$$

A homogeneous medium is one for which the quantities ϵ , μ & σ are constant throughout the medium.

* Wave Equations for a conducting Medium :

The application of Maxwell's equation is the prediction of existence of electromagnetic wave. Electromagnetic wave equation can be obtained from Maxwell's equations.

The Maxwell's equation from Ampere's law in point form is

$$\begin{aligned} \nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ &= \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \quad \text{--- (1)} \end{aligned}$$

The Maxwell's equation from Faraday's law in point form is given by,

$$\begin{aligned} \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ &= -\mu \frac{\partial \vec{H}}{\partial t} \quad \text{--- (2)} \end{aligned}$$

Taking curl on both sides

$$\nabla \times \nabla \times \vec{E} = -\mu \nabla \times \frac{\partial \vec{H}}{\partial t} \quad \text{--- (3)}$$

Differentiating eq - (2) on both sides

$$\nabla \times \frac{\partial \vec{H}}{\partial t} = \frac{\partial \nabla \times \vec{H}}{\partial t}$$

$$\frac{\partial}{\partial t} \left[\sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t} \right]$$

$$\Rightarrow \nabla \times \frac{\partial \vec{H}}{\partial t} = \sigma \frac{\partial \vec{E}}{\partial t} + \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{--- (4)}$$

Substituting eq. (4) in eq. (3)

$$\nabla \times \nabla \times \vec{E} = -\mu \left[\sigma \frac{\partial \vec{E}}{\partial t} + \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \right] \quad \text{--- (5)}$$

According to the Identity,

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \quad \text{--- (6)}$$

But,

$$\nabla \cdot \vec{E} = \frac{1}{\epsilon} \cdot \nabla \cdot \vec{D}$$

Since, there is no net charge within the conductor, the charge density $\rho = 0$

$$\nabla \cdot \vec{D} = 0$$

$$\nabla \cdot \vec{E} = 0$$

The eq. (6) becomes

$$\nabla \times \nabla \times \vec{E} = -\nabla^2 \vec{E} \quad \text{--- (7)}$$

Comparing equation (5) & (7)

$$\nabla^2 \vec{E} = \mu \sigma \frac{\partial \vec{E}}{\partial t} + \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \boxed{\nabla^2 \vec{E} - \mu \sigma \frac{\partial \vec{E}}{\partial t} - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0} \quad \text{--- (8)}$$

This is the wave equation in terms of field $\vec{E}$.

The Maxwell's Equation from Ampere's law in point form is given by.

$$\nabla \times \vec{H} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}$$

Taking curl on both sides

$$\nabla \times \nabla \times \vec{H} = \sigma \nabla \times \vec{E} + \epsilon \nabla \times \frac{\partial \vec{E}}{\partial t} \quad \text{--- (9)}$$

3

The maxwell's equation from faradays law is

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \text{--- (10)}$$

Differentiating w.r.t 't' the above eqⁿ becomes

$$\nabla \times \frac{\partial \mathbf{E}}{\partial t} = -\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} \quad \text{--- (11)}$$

Substituting eq. (8) 10 & 11 in eq. (9)

$$\nabla \times \nabla \times \mathbf{H} = -\mu \sigma \frac{\partial \mathbf{H}}{\partial t} - \mu \epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} \quad \text{--- (12)}$$

According to identity

$$\nabla \times \nabla \times \mathbf{H} = \nabla(\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H}$$

$$\text{But } \nabla \cdot \mathbf{B} = \mu \nabla \cdot \mathbf{H} = 0$$

$$\text{Then } \nabla \times \nabla \times \mathbf{H} = -\nabla^2 \mathbf{H} \quad \text{--- (13)}$$

Comparing equations (12) & (13)

$$\nabla^2 \mathbf{H} = \mu \sigma \frac{\partial \mathbf{H}}{\partial t} + \mu \epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

$$\Rightarrow \boxed{\nabla^2 \mathbf{H} - \mu \sigma \frac{\partial \mathbf{H}}{\partial t} - \mu \epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0} \quad \text{--- (14)}$$

This is the wave equation in terms of magnetic field $\mathbf{H}$.

* Wave Equation for Free Space (Dielectric Medium) :-

For Free Space (dielectric medium) the conductivity of the medium is zero (i.e. $\sigma = 0$) and there is no charge containing in it (i.e. $\rho = 0$).

The electromagnetic wave equations for free space can be obtained from Maxwell's equations.

$$\nabla \times H = \frac{\partial D}{\partial t}$$

$$\nabla \times E = \frac{\partial B}{\partial t}$$

$$\nabla \cdot D = 0$$

$$\nabla \cdot B = 0$$

The Maxwell's equation from Faraday's law for free space in point form is

$$\begin{aligned}\nabla \times E &= -\frac{\partial B}{\partial t} \\ &= -\mu \frac{\partial H}{\partial t}\end{aligned}$$

Taking curl on Both sides.

$$\nabla \times \nabla \times E = -\mu \nabla \times \frac{\partial H}{\partial t} \quad \text{--- (1)}$$

The Maxwell's equation from Ampere's law for the free space in point form is

$$\begin{aligned}\nabla \times H &= \frac{\partial D}{\partial t} \\ &= \epsilon \frac{\partial E}{\partial t}\end{aligned}$$

Differentiating wrt 't' on Both sides

$$\nabla \times \frac{\partial H}{\partial t} = \epsilon \frac{\partial^2 E}{\partial t^2} \quad \text{--- (2)}$$

Substituting eq-2 in eq-1

$$\nabla \times \nabla \times E = -\mu \epsilon \frac{\partial^2 E}{\partial t^2} \quad \text{--- (3)}$$

According to identity

$$\nabla \times \nabla \times E = \nabla (\nabla \cdot E) - \nabla^2 E$$

$$\text{But } \nabla \cdot E = \frac{1}{\epsilon} \nabla \cdot D = \frac{\rho}{\epsilon} = 0$$

$$\text{Then } \nabla \times \nabla \times E = -\nabla^2 E \quad \text{--- (4)}$$

Comparing (3)eq & (4)eq

$$\nabla^2 E = \mu \epsilon \frac{\partial^2 E}{\partial t^2}$$

5

$$\boxed{\nabla^2 E - \mu \epsilon \frac{\partial^2 E}{\partial t^2} = 0} \quad \text{--- (5)}$$

This is the wave equation for free space in terms of electric field.

The Maxwell's equation from Ampere's law for free space in point form is given by.

$$\nabla \times H = \epsilon \frac{\partial E}{\partial t}$$

Taking curl on Both sides

$$\nabla \times \nabla \times H = \epsilon \nabla \times \frac{\partial E}{\partial t} \quad \text{--- (6)}$$

The Maxwell's equation from Faraday's Law is

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

Differentiating on Both sides

$$\nabla \times \frac{\partial E}{\partial t} = -\mu \frac{\partial^2 H}{\partial t^2} \quad \text{--- (7)}$$

Substituting eq- (7) in (6)

$$\nabla \times \nabla \times H = -\mu \epsilon \frac{\partial^2 H}{\partial t^2} \quad \text{--- (8)}$$

According to the identity,

$$\nabla \times \nabla \times H = \nabla (\nabla \cdot H) - \nabla^2 H$$

$$\text{But } \nabla \cdot H = \frac{1}{\mu} \nabla \cdot B = 0$$

$$\therefore \nabla \times \nabla \times H = -\nabla^2 H \quad \text{--- (9)}$$

Comparing equations (8) & (9)

$$\nabla^2 H = \mu \epsilon \frac{\partial^2 H}{\partial t^2}$$

$$\boxed{\nabla^2 H - \mu \epsilon \frac{\partial^2 H}{\partial t^2} = 0} \quad \text{--- (10)}$$

This is the wave equation for free space in terms of H.

Uniform plane wave :-

If the plane of a wave is the same for all points on a plane surface is called "plane wave".

If the amplitude is also constant in a plane wave is called "Uniform plane wave".

The properties uniform plane waves are given as follows

- ①. At Every point in space electric field (E) and magnetic field (H) are perpendicular to each other and to the direction of travel.
- ②. The fields vary harmonically with time and at the same frequency, everywhere in space.
- ③. Each field has the same direction, magnitudes and phase at every point in any plane perpendicular to the direction of wave travel.

If the electric field is in x-direction and the magnetic field in y-direction, then the wave is travelling in z-direction.

The plane wave equation

$$\frac{\partial^2 E}{\partial x^2} = \mu \epsilon \frac{\partial^2 E}{\partial t^2}$$

$$\frac{\partial^2 E_y}{\partial x^2} = \mu \epsilon \frac{\partial^2 E_y}{\partial t^2}$$

$$\frac{\partial^2 E_z}{\partial x^2} = \mu \epsilon \frac{\partial^2 E_z}{\partial t^2}$$

In a region in which there is no charge density

$$\nabla \cdot E = \frac{1}{\epsilon} \nabla \cdot D = 0$$

$$\text{i.e.} \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

For a uniform plane wave in which E is independent

of y and z , the last two terms of this relation are equal to zero so that it reduces to

$$\frac{\partial E_x}{\partial x} = 0$$

This equation shows that there is no variation of E_x in the x -direction

Differentiating the above equation

$$\frac{\partial^2 E_x}{\partial x^2} = 0$$

It requires that either E_x be zero or constant. A field is not satisfying if E_x is constant and so E_x must be zero. Therefore a uniform plane wave propagating (travelling) in the x -direction has no x -component of E .

$$\nabla \cdot B = \nabla \cdot \mu \cdot H = \nabla \cdot H = 0$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0$$

Since H is propagating in y -direction, it is independent of x and z

$$\text{Then } \frac{\partial H_x}{\partial y} = 0, \quad \frac{\partial^2 H_x}{\partial y^2} = 0$$

Since, H_y is not a constant

$\rightarrow H_y$ must be zero

$\therefore H_y = 0$ for uniform plane wave.

Therefore, the uniform plane electromagnetic waves are transverse and have components of E and H only in directions perpendicular to the direction of propagation.

④ RELATION BETWEEN E AND H IN UNIFORM PLANE WAVE:

For a uniform plane wave travelling in the x -direction E and H are both independent of y and z , and E, H have no x -component.

$$\text{In this case, } \nabla \times E = -\frac{\partial E_x}{\partial x} \hat{y} + \frac{\partial E_y}{\partial x} \hat{z} = -\epsilon \frac{\partial E}{\partial t}$$

$$\nabla \times H = -\frac{\partial H_x}{\partial x} \hat{y} + \frac{\partial H_y}{\partial x} \hat{z} = \mu \frac{\partial H}{\partial t}$$

First Maxwell Equation can be written as,

$$-\frac{\partial H_x}{\partial x} \hat{y} + \frac{\partial H_y}{\partial x} \hat{z} = \epsilon \left(\frac{\partial E_y}{\partial t} \hat{y} - \frac{\partial E_z}{\partial t} \hat{z} \right)$$

Second Equation becomes,

$$-\frac{\partial E_z}{\partial x} \hat{y} + \frac{\partial E_y}{\partial x} \hat{z} = -\mu \left(\frac{\partial H_y}{\partial t} \hat{y} + \frac{\partial H_z}{\partial t} \hat{z} \right)$$

Equating the $\hat{y}$ terms and then the $\hat{z}$ term yields the four relations

$$-\frac{\partial H_z}{\partial x} = \epsilon \frac{\partial E_y}{\partial t} \quad \text{--- (1)}$$

$$\frac{\partial H_y}{\partial x} = \epsilon \frac{\partial E_z}{\partial t} \quad \text{--- (2)}$$

$$\frac{\partial E_z}{\partial x} = \mu \frac{\partial H_y}{\partial t} \quad \text{--- (3)}$$

$$\frac{\partial E_y}{\partial x} = -\mu \frac{\partial H_z}{\partial t} \quad \text{--- (4)}$$

Now, if $E_y = f_1(x - v_0 t)$, where, $v_0 = \frac{1}{\sqrt{\mu \epsilon}}$

$$\begin{aligned} \text{then } \frac{\partial E_y}{\partial t} &= \frac{\partial f_1}{\partial (x - v_0 t)} \cdot \frac{\partial (x - v_0 t)}{\partial t} \\ &= -v_0 \cdot \frac{\partial f_1}{\partial (x - v_0 t)} \end{aligned}$$

This is generally written as

$$\frac{\partial E_y}{\partial t} = f_1'(x - v_0 t) \frac{\partial (x - v_0 t)}{\partial t} = -v_0 f_1'(x - v_0 t)$$

Where $f_1'(x - V_0 t)$, or more simply f_1' means

$$\frac{\partial f_1(x - V_0 t)}{\partial (x - V_0 t)}$$

Substituting for $\frac{\partial E_y}{\partial t}$ in eq. (1) above gives

$$\frac{\partial H_z}{\partial x} = V_0 \epsilon f_1'$$

$$\Rightarrow H_z = \sqrt{\frac{\epsilon}{\mu}} \int f_1' dx + C$$

$$\text{Now, } \frac{\partial f_1}{\partial x} = f_1' \cdot \frac{\partial (x - V_0 t)}{\partial x} = f_1'$$

$$\text{Hence } H_z = \sqrt{\frac{\epsilon}{\mu}} \int \frac{\partial f_1}{\partial x} dx + C$$

$$= \sqrt{\frac{\epsilon}{\mu}} f_1 + C = \sqrt{\frac{\epsilon}{\mu}} E_y + C \quad \text{--- (5)}$$

$$H_z = \sqrt{\frac{\epsilon}{\mu}} E_y$$

$$\frac{E_y}{H_z} = \sqrt{\frac{\mu}{\epsilon}} \quad \text{--- (6)}$$

Similarly it can be shown that

$$\frac{E_z}{H_y} = -\sqrt{\frac{\mu}{\epsilon}}$$

$$\text{Since, } E = \sqrt{E_y^2 + E_z^2} \text{ and } H = \sqrt{H_z^2 + H_y^2}$$

where E and H are the total electric and magnetic field strengths, these also results

$$\frac{E}{H} = \sqrt{\frac{\mu}{\epsilon}} \quad \text{--- (8)}$$

Since, the Ratio $\frac{E}{H} = \sqrt{\frac{\mu}{\epsilon}}$, units of E are volts per meter and H are ampere's per meter. Ratio will be having the dimension of impedance and ohms.

* Sinusoidal Time Variations:

Most generators produce voltages and currents and hence electric and magnetic fields, which vary sinusoidally with time. So, it is customary in most problems to assume sinusoidal variations.

This can be expressed in terms by writing, for example,

$$E = E_0 \cos \omega t$$

$$E = E_0 \sin \omega t$$

where $f = \omega/2\pi$ is the frequency of the variation.

Consider the electric field strength vector,

The time varying field $\vec{E}(r, t)$ may be expressed in terms of corresponding phasor quantity.

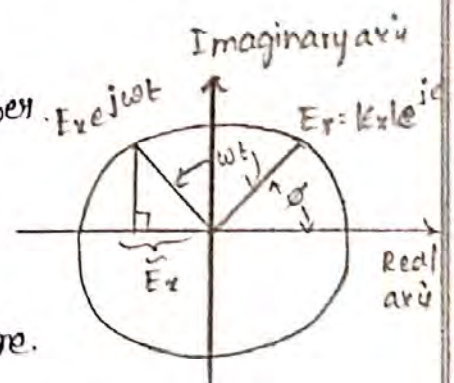
$$\tilde{E}(r, t) = \text{Re} \{ E(r) e^{j\omega t} \}$$

in which $(\sim)$ has been placed over time-varying quantity to distinguish it from phasor quantity.

The phasor E_x is defined by the relation

$$\tilde{E}_x(r, t) = \text{Re} \{ E_x(r) e^{j\omega t} \}$$

In general, $E_x(r)$ is a complex number. $E_x e^{j\omega t}$ and thus at some fixed point r in space E_x may be represented as a point in complex plane shown in figure.



The phase of the sinusoidal is determined by the argument ϕ of the complex number E_x and thus the time varying quantity can be expressed as

$$\begin{aligned} \tilde{E}_x &= \text{Re} \{ |E_x| e^{j\phi} e^{j\omega t} \} \\ &= |E_x| \cos(\omega t + \phi) \end{aligned}$$

This is one of the possible phasor definition.

* Maxwell's Equations Using Phasor Notation:

Consider for instance the equation given in time varying form by.

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J} \quad \text{--- (1)}$$

For the sinusoidal steady state we may substitute the phasor relation as follows:

$$\nabla \times \text{Re}\{\vec{H}e^{j\omega t}\} = \frac{\partial}{\partial t} \text{Re}\{\vec{D}e^{j\omega t}\} + \text{Re}\{\vec{J}e^{j\omega t}\} \quad \text{--- (2)}$$

The operations of taking the real part and differentiation may be interchanged,

$$\text{Re}\{(\nabla \times \vec{H} - j\omega \vec{D} - \vec{J})e^{j\omega t}\} = 0$$

$$\Rightarrow \nabla \times \vec{H} = j\omega \vec{D} + \vec{J} \quad \text{--- (3)}$$

For sinusoidal time variations, Maxwell's equation may be expressed in phasor form as

$$\nabla \times \vec{H} = j\omega \vec{D} + \vec{J}$$

$$\oint \vec{H} \cdot d\vec{s} = \int (j\omega \vec{D} + \vec{J}) \cdot d\vec{a}$$

$$\nabla \times \vec{E} = -j\omega \vec{B}$$

$$\oint \vec{E} \cdot d\vec{s} = -\int j\omega \vec{B} \cdot d\vec{a}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\oint \vec{D} \cdot d\vec{a} = \int \rho dv$$

$$\nabla \cdot \vec{B} = 0$$

$$\oint \vec{B} \cdot d\vec{a} = 0$$

The above Equations contain the equation of continuity,

$$\nabla \cdot \vec{J} = -j\omega \rho \quad \text{or} \quad \oint \vec{J} \cdot d\vec{a} = -\int j\omega \rho dv$$

The constitutive relations retain their form, being

$$\vec{D} = \epsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

$$\vec{J} = \sigma \vec{E}$$

For sinusoidal time variations the wave Equation for electric field in lossless medium becomes,

$$\nabla^2 E = -\omega^2 \mu \epsilon E$$

which is the vector Helmholtz equation.

→ The Wave Equation in a conducting medium becomes

$$\nabla^2 E + (\omega^2 \mu \epsilon - j\omega \mu \sigma) E = 0$$

The Equation of same form may be written for H.

⑧ Wave Propagation in Lossless Medium :-

The wave equation is

$$\nabla^2 E = \mu \epsilon \frac{\partial^2 E}{\partial t^2}$$

The phasor value of E is

$$E(x, t) = \text{Re} \{ E(x) e^{j\omega t} \}$$

Applying to the wave equation.

$$\nabla^2 \text{Re} \{ E e^{j\omega t} \} = \mu \epsilon \frac{\partial^2}{\partial t^2} \text{Re} \{ E e^{j\omega t} \}$$

$$\Rightarrow \nabla^2 \text{Re} \{ E e^{j\omega t} \} = \mu \epsilon \text{Re} \{ -\omega^2 E e^{j\omega t} \}$$

$$\Rightarrow \text{Re} \{ (\nabla^2 E + \mu \epsilon \omega^2 E) e^{j\omega t} \} = 0$$

$$\Rightarrow \boxed{\nabla^2 E + \mu \epsilon \omega^2 E = 0}$$

This is the Wave Equation for lossless Medium in phasor form and it is called vector Helmholtz equation.

$$\nabla^2 E + \beta^2 E = 0$$

$$\text{where } \beta^2 = \mu \epsilon \omega^2$$

$$\beta = \sqrt{\mu \epsilon} \omega$$

"β" is called Phase Shift Constant

The velocity of propagation is,

$$v = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\mu \epsilon} \omega} = \frac{1}{\sqrt{\mu \epsilon}}$$

The Wave propagates in x-direction i.e. no variation in y & z.

$$\frac{\partial^2 E}{\partial x^2} + \beta^2 E = 0$$

The solution of the Equation is $E = C_1 e^{-j\beta x} + C_2 e^{j\beta x}$.

WAVE PROPAGATION IN A CONDUCTING MEDIUM:-

The wave equation for conducting medium is,

$$\nabla^2 E - \mu \epsilon \frac{\partial^2 E}{\partial t^2} - \mu \sigma \frac{\partial E}{\partial t} = 0$$

The phasor form of wave equation is,

$$\nabla^2 E + \mu \epsilon \omega^2 E - j\omega \mu \sigma E = 0$$

$$\Rightarrow \nabla^2 E - j^2 \mu \epsilon \omega^2 E - j\omega \mu \sigma E = 0$$

$$\Rightarrow \nabla^2 E - j(\omega \mu \sigma E + j \mu \epsilon \omega^2 E) = 0$$

$$\Rightarrow \nabla^2 E - j\omega \mu (\sigma + j\omega \epsilon) E = 0$$

$$\Rightarrow \nabla^2 E - \gamma^2 E = 0$$

where $\gamma^2 = j\omega \mu (\sigma + j\omega \epsilon)$

" γ " is called propagation constant, which has both real and imaginary parts.

$$\gamma = \alpha + j\beta$$

where ' α ' is attenuation constant

' β ' is phase constant

$$\gamma = \alpha + j\beta$$

$$= \sqrt{j\omega \mu (\sigma + j\omega \epsilon)}$$

$$\Rightarrow \alpha + j\beta = \sqrt{j\omega \mu (\sigma + j\omega \epsilon)}$$

$$\text{S.O.B.S}$$

$$\alpha^2 - \beta^2 + 2j\alpha\beta = j\omega \mu (\sigma + j\omega \epsilon)$$

$$\Rightarrow \alpha^2 - \beta^2 + 2j\alpha\beta = j\omega \mu \sigma - \omega^2 \mu \epsilon$$

Equating Real & Imaginary parts

$$\alpha^2 - \beta^2 = -\omega^2 \mu \epsilon$$

$$2\alpha\beta = \omega \mu \sigma$$

$$\alpha^2 + \beta^2 = \sqrt{(\alpha^2 - \beta^2)^2 + 4\alpha^2\beta^2}$$

$$= \sqrt{(-\omega^2 \mu \epsilon)^2 + (\omega \mu \sigma)^2}$$

$$\Rightarrow \alpha^2 + \beta^2 = \sqrt{\omega^4 \mu^2 \epsilon^2 + \omega^2 \mu^2 \sigma^2}$$

$$\alpha^2 - \beta^2 = -\omega^2 \mu \epsilon$$

Adding these two equations

$$2\alpha^2 = -\omega^2 \mu \epsilon + \sqrt{\omega^4 \mu^2 \epsilon^2 + \omega^2 \mu^2 \sigma^2}$$

$$\alpha^2 = \frac{-\omega^2 \mu \epsilon}{2} + \frac{1}{2} \sqrt{\omega^4 \mu^2 (\omega^2 \epsilon^2 + \sigma^2)}$$

$$= \frac{-\omega^2 \mu \epsilon}{2} + \frac{\omega \cdot \mu}{2} \sqrt{\omega^2 \epsilon^2 \left(1 + \frac{\sigma^2}{\omega^2 \epsilon^2}\right)}$$

$$= \frac{-\omega^2 \mu \epsilon}{2} + \frac{\omega^2 \mu \epsilon}{2} \sqrt{1 + \frac{\sigma^2}{\omega^2 \epsilon^2}}$$

$$\alpha^2 = \frac{\omega^2 \mu \epsilon}{2} \left[\sqrt{1 + \frac{\sigma^2}{\omega^2 \epsilon^2}} - 1 \right]$$

$$\alpha = \sqrt{\frac{\omega^2 \mu \epsilon}{2} \left[\sqrt{1 + \frac{\sigma^2}{\omega^2 \epsilon^2}} - 1 \right]}$$

By subtracting $\alpha^2 - \beta^2$ from $\alpha^2 + \beta^2$, the value of " β " becomes

$$2\beta^2 = \sqrt{\omega^4 \mu^2 \epsilon^2 + \omega^2 \mu^2 \sigma^2} + \omega^2 \mu \epsilon$$

$$\Rightarrow \beta = \omega \sqrt{\frac{\mu \epsilon}{2} \left[\sqrt{1 + \frac{\sigma^2}{\omega^2 \epsilon^2}} + 1 \right]}$$

CONDUCTORS AND DIELECTRICS :-

In electromagnetics, materials are divided into 2 classes: Conductors and dielectrics or insulators.

As some media behave like good conductors at one frequency range and like good dielectrics at some other frequency range. There is no exact dividing line between conductors and dielectrics.

In Maxwell's first equation:

$$\nabla \times H = \sigma E + j\omega \epsilon E$$

where. " σE " is conduction current density

" $\omega \epsilon E$ " is displacement current density.

$$\text{ie. --- } \frac{\sigma E}{\omega \epsilon E} = \frac{\sigma}{\omega \epsilon}$$

where ' σ ' is conductivity of the medium

' ϵ ' is permittivity of the medium

' ω ' is Angular frequency.

The displacement current density,

$$J_d = D = \epsilon E = j\omega \epsilon E$$

The conduction current density,

$$J_c = \sigma E.$$

$$\therefore \boxed{\frac{J_c}{J_d} = \frac{\sigma}{\omega \epsilon}} \Rightarrow \begin{array}{l} \text{For copper, } \frac{\sigma}{\omega \epsilon} = 3.5 \times 10^8 \\ \text{For Mica, } \frac{\sigma}{\omega \epsilon} = 2 \times 10^{-4} \end{array}$$

Definition of a good conductor,

If $\frac{\sigma}{\omega \epsilon} \gg 1$, the medium is a good conductor

Definition of a good dielectric,

If $\frac{\sigma}{\omega \epsilon} \ll 1$, the medium is a good dielectric.

Dissipation factor of a dielectric material (D_f)

$$\boxed{D_f = \frac{\sigma}{\omega \epsilon}}$$

When D_f is small for a specific type of dielectric material, the dissipation factor is practically the same as its power factor which is equal to " $\sin \phi$ ".

$$\text{Here. } PF = \sin \phi$$

$$D_f = \sin \phi$$

$$\boxed{\phi = \tan^{-1} D_f} //$$

WAVE PROPAGATION CHARACTERISTICS IN GOOD DIELECTRICS :-

The wave equation

For Dielectrics $\frac{\sigma}{\omega \epsilon} \ll 1$.

$$\sqrt{1 + \frac{\sigma^2}{\omega^2 \epsilon^2}} = \left(1 + \frac{\sigma^2}{\omega^2 \epsilon^2}\right)^{1/2}$$

$$= \left(1 + \frac{\sigma^2}{2\omega^2 \epsilon^2}\right)$$

The Attenuation factor (α) is given by,

$$\alpha = \omega \sqrt{\frac{\mu \epsilon}{2}} \left(\sqrt{1 + \frac{\sigma^2}{\omega^2 \epsilon^2}} - 1\right)$$

$$= \omega \sqrt{\frac{\mu \epsilon}{2}} \left(1 + \frac{\sigma^2}{2\omega^2 \epsilon^2} - 1\right)$$

$$= \omega \sqrt{\frac{\mu \epsilon}{2}} \left(\frac{\sigma^2}{2\omega^2 \epsilon^2}\right)$$

$$= \omega \sqrt{\frac{\mu \sigma^2}{4\omega^2 \epsilon}}$$

$$\boxed{\alpha = \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}}}$$

The phase shift constant (β) is given by,

$$\beta = \omega \sqrt{\frac{\mu \epsilon}{2}} \left[\sqrt{1 + \frac{\sigma^2}{\omega^2 \epsilon^2}} + 1\right]$$

$$= \omega \sqrt{\frac{\mu \epsilon}{2}} \left(2 + \frac{\sigma^2}{2\omega^2 \epsilon^2}\right)$$

$$= \omega \sqrt{\mu \epsilon} \left(1 + \frac{\sigma^2}{4\omega^2 \epsilon^2}\right)$$

$$= \omega \sqrt{\mu \epsilon} \left(1 + \frac{\sigma^2}{4\omega^2 \epsilon^2}\right)^{1/2}$$

$$\boxed{\beta \approx \omega \sqrt{\mu \epsilon} \left(1 + \frac{\sigma^2}{8\omega^2 \epsilon^2}\right)}$$

The velocity of the wave in the dielectric is,

$$v = \frac{\omega}{\beta} = \frac{\omega}{\omega \sqrt{\mu \epsilon} \left(1 + \frac{\sigma^2}{8\omega^2 \epsilon^2}\right)}$$

$$\approx \frac{1}{\sqrt{\mu \epsilon} \left(1 - \frac{\sigma^2}{8\omega^2 \epsilon^2}\right)}$$

$$\Rightarrow \boxed{v = v_0 \left(1 - \frac{\sigma^2}{8\omega^2 \epsilon^2}\right)} \quad \left(\because v_0 = \frac{1}{\sqrt{\mu \epsilon}}\right)$$

Intrinsic or characteristic impedance of medium is given by,

$$\eta = \sqrt{\frac{j\omega \mu}{\sigma + j\omega \epsilon}}$$

$$= \sqrt{\frac{j\omega \mu}{j\omega \epsilon \left(1 + \frac{\sigma}{j\omega \epsilon}\right)}}$$

$$= \sqrt{\frac{\mu}{\epsilon} \left(1 + \frac{\sigma}{j\omega \epsilon}\right)^{-1}}$$

$$= \sqrt{\frac{\mu}{\epsilon} \left(1 - \frac{\sigma}{j\omega \epsilon}\right)}$$

$$= \sqrt{\frac{\mu}{\epsilon} \left(1 + \frac{j\sigma}{\omega \epsilon}\right)}$$

$$= \sqrt{\frac{\mu}{\epsilon} \left(1 + \frac{j\sigma}{\omega \epsilon}\right)^{1/2}}$$

$$\therefore \boxed{\eta = \sqrt{\frac{\mu}{\epsilon} \left(1 + \frac{j\sigma}{\omega \epsilon}\right)}}$$

Wave propagation in Good conductor:-

For good conductor $\frac{\sigma}{\omega \epsilon} \gg 1$

$$\delta = \sqrt{j\omega \mu (\sigma + j\omega \epsilon)}$$

$$= \sqrt{j\omega \mu \sigma \left(1 + \frac{j\omega \epsilon}{\sigma}\right)}$$

$$= \sqrt{j\omega \mu \sigma}$$

$$\left(\because \frac{\omega \epsilon}{\sigma} \ll 1\right)$$

$$\gamma = \sqrt{\omega \mu \sigma} \angle 45^\circ$$

$$= \sqrt{\omega \mu \sigma} \left(\frac{1}{2} + j \cdot \frac{1}{2} \right)$$

$$\therefore \boxed{\alpha = \beta = \sqrt{\frac{\omega \mu \sigma}{2}}}$$

The velocity of the wave in conductor

$$v = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\frac{\omega \mu \sigma}{2}}}$$

$$\boxed{v = \sqrt{\frac{2\omega}{\mu \sigma}}}$$

The intrinsic impedance of the conductor

$$\eta = \sqrt{\frac{j\omega\mu}{j\omega\epsilon \left(1 + \frac{\sigma}{j\omega\epsilon}\right)}}$$

$$= \sqrt{\frac{j\omega\mu}{j\omega\epsilon \cdot \frac{\sigma}{j\omega\epsilon}}} = \sqrt{\frac{j\omega\mu}{\sigma}}$$

$$\boxed{\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ}$$

It is found that in good conductors, α and β are large since ' σ ' is large i.e. the wave is attenuated greatly as it progresses through the conductor. But the velocity and characteristic function impedance are reduced considerably by

④. Depth of Penetration :

Definition : The depth of penetration is defined as that depth at which the wave attenuates to $\frac{1}{e}$ or approximately 37% of its original amplitude. It is also called as Skin Depth.

The amplitude of the wave decreases by the factor $e^{-\alpha x}$ as it propagates through a distance 'x'.

By Definition $e^{-\alpha x} = \frac{1}{e}$

If $x = \delta$

$$e^{-\alpha \delta} = \frac{1}{e} = e^{-1}$$

$$\alpha \delta = 1$$

$$\Rightarrow \boxed{\delta = \frac{1}{\alpha}}$$

This is the depth of penetration or skin depth.

$$\delta = \frac{1}{\alpha}$$

$$= \frac{1}{\omega \sqrt{\frac{\mu \epsilon}{2} (\sqrt{1 + \frac{\sigma^2}{\omega^2 \epsilon^2}} - 1)}}$$

For a good Conductor the Depth of Penetration δ

$$\boxed{\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega \mu \sigma}}} //$$

Polarisation :-

Polarisation of a wave is defined as the direction of the electric field at a given point as a function of time.

The polarisation of a composite wave is the direction of the electric field.

Types of polarisations

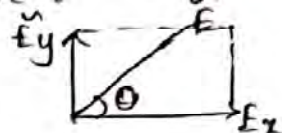
There are three types, namely.

- a. Linear polarisation
- b. Circular polarisation
- c. Elliptical polarisation.

* Linear Polarisation :-

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If both $\tilde{E}_x$ and $\tilde{E}_y$ are present and are in phase, i.e., resultant electric field has a direction dependent on the relative magnitude of $\tilde{E}_x$ and $\tilde{E}_y$. The angle which this direction makes with the x -axis is $\tan^{-1}\left(\frac{\tilde{E}_y}{\tilde{E}_x}\right)$ and this angle will be constant with time as shown in fig. If the direction of resultant vector is constant with time, the wave is said to be linearly polarised.



* Elliptical Polarisation :-

If $\tilde{E}_x$ and $\tilde{E}_y$ are not equal in magnitude and they differ by 90° phase, then the tip of the resultant electric vector traces out an ellipse. The wave is said to be elliptically polarised.

∴ Here E_c can be written as,

$$E_c = a \cos \omega t + j b \sin \omega t \quad \text{where } a \text{ \& } b \text{ are constants}$$

$$\tilde{E}(0, t) = a \cos \omega t \hat{a}_x - b \sin \omega t \hat{a}_y$$

The components of the time varying field are

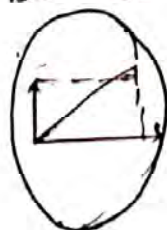
$$\tilde{E}_x = a \cos \omega t$$

$$\tilde{E}_y = -b \sin \omega t$$

Then

$$\boxed{\frac{\tilde{E}_x^2}{a^2} + \frac{\tilde{E}_y^2}{b^2} = 1}$$

This is the equation of an ellipse and hence the wave is said to be elliptically polarised.



Circular Polarisation:

If $\tilde{E}_x$ and $\tilde{E}_y$ have equal magnitudes and a 90° phase difference, the locus of the resultant $\tilde{E}$ is a circle and the wave is circularly polarised.

If the y component leads the x-component by 90° and if both components have amplitude E_a , the electric field at $z=0$ is given by the complex vector:

$$\mathbf{E}_0 = (\hat{x} + j\hat{y}) E_a$$

The corresponding time-varying field is given by

$$\tilde{\mathbf{E}}(0, t) = (\hat{x} \cos \omega t - \hat{y} \sin \omega t) E_a$$

The components are

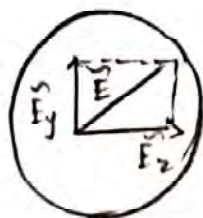
$$\tilde{E}_x = E_a \cos \omega t$$

$$\tilde{E}_y = E_a \sin \omega t$$

$\therefore$ These components satisfy the relation

$$\boxed{\tilde{E}_x^2 + \tilde{E}_y^2 = E_a^2}$$

This equation shows that the locus of the resultant $\tilde{E}$ is a circle whose radius is E_a .



* WAVE PROPAGATION IN LOSSLESS MEDIUM :

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Summary :

1. Phase shift constant (β): defined as measure of the phase shift in radians per unit length.
2. Wave length of the wave (λ): defined as that distance through which sinusoidal wave passes through cycle of 2π radians i.e. $\lambda = \frac{2\pi}{\beta}$
3. Phase velocity (V_p): defined as the velocity of some point in sinusoidal waveform.
4. Intrinsic or characteristic impedance: It has a finite value of conductivity is given by $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$

* WAVE PROPAGATION IN FREE SPACE :

Summary :

1. Propagation constant, $\gamma = j\omega\sqrt{\mu_0\epsilon_0}$ (m^{-1})
2. Phase shift constant, $\beta = \omega\sqrt{\mu_0\epsilon_0}$ (rad/m)
3. Velocity of propagation, $V_0 = \frac{1}{\sqrt{\mu_0\epsilon_0}}$ (m/s)
4. $\frac{E}{H} = 120\pi\Omega = \sqrt{\frac{\mu_0}{\epsilon_0}}$ (Ω)
5. Attenuation constant (α) = 0.

* WAVE PROPAGATION IN CONDUCTING MEDIUM :

Summary :

1. Propagation constant, $\gamma = \sqrt{-\omega^2\mu\epsilon + j\omega\mu\sigma}$, (m^{-1})
2. Phase shift constant, $\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} + 1\right)}$, (rad/m)
3. Attenuation constant, $\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} - 1\right)}$, (dB/m)
4. Velocity of propagation, $v = f\lambda = \frac{\omega}{\beta} = \frac{1}{\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} + 1\right)}}$, (m/s)

$$5. \lambda = \frac{2\pi}{\beta} \text{ (m)} \text{ (or)} \lambda = \frac{1}{\frac{\omega}{c} \sqrt{\mu\epsilon} \left(\sqrt{1 + \frac{\sigma^2}{\omega^2\epsilon^2}} + 1 \right)}$$

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$$6. \text{ Intrinsic Impedance, } \eta = \frac{E}{H} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \text{ } (\Omega)$$

*

WAVE PROPAGATION IN GOOD DIELECTRICS:

Summary:

$$1. \text{ Attenuation constant, } \alpha = \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}}, \text{ (dB/m)}$$

$$2. \text{ Phase shift constant, } \beta = \omega \sqrt{\mu\epsilon} \left(1 + \frac{\sigma^2}{8\omega^2\epsilon^2} \right), \text{ (rad/m)}$$

$$3. \text{ Velocity of wave, } v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon} \left(1 - \frac{\sigma^2}{8\omega^2\epsilon^2} \right)}, \text{ (m/s)}$$

$$4. \text{ Intrinsic Impedance, } \eta = \sqrt{\frac{\mu}{\epsilon}} \left(1 + j \cdot \frac{\sigma}{2\omega\epsilon} \right), \text{ } (\Omega)$$

*

WAVE PROPAGATION IN GOOD CONDUCTORS:

Summary:

$$1. \alpha = \sqrt{\frac{\omega\mu\sigma}{2}}, \text{ (dB/m)}$$

$$2. \beta = \sqrt{\frac{\omega\mu\sigma}{2}}, \text{ (rad/m)}$$

$$3. v = \sqrt{\frac{2\omega}{\mu\sigma}}, \text{ (m/s)}$$

$$4. \eta = \sqrt{\frac{j\omega\mu}{\sigma}}, \text{ } (\Omega)$$

Problem :

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If $\epsilon_r = 9$, $\mu = \mu_0$ for the medium in which a wave with a frequency, $f = 0.3 \text{ GHz}$ is propagating, determine the propagation constant and intrinsic impedance of the medium when $\sigma = 0$

Sol :

The expression for propagation constant,

$$\begin{aligned}\gamma &= \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} \\ &= j\omega\sqrt{\mu\epsilon} \quad \text{for } \sigma = 0\end{aligned}$$

$$\mu_r = 1, \quad \epsilon_r = 9, \quad \sigma = 0, \quad f = 0.3 \text{ GHz}$$

$$\begin{aligned}\gamma &= j2\pi \times 3 \times 10^8 \times \sqrt{4\pi \times 10^{-7} \times 9 \times 8.854 \times 10^{-12}} \\ &= j6\pi \times 10^8 \times 2 \times 3 \sqrt{8.854 \times 10^{-19} \times \pi} \\ &= j36\pi \times 10^8 \times 10^{-9} \times 1.66780\end{aligned}$$

$$\boxed{\gamma = j18.8624} \text{ (1/m)}$$

Intrinsic Impedance,

$$\begin{aligned}\eta &= \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = \sqrt{\frac{\mu}{\epsilon}} \\ &= \sqrt{\frac{\mu_0}{9\epsilon_0}} \\ &= \frac{1}{3} \sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{1}{3} \times 120\pi\end{aligned}$$

$$\boxed{\eta = 40\pi \Omega} //$$

Reflection by a Perfect Conductor - Normal Incidence :-

When an electromagnetic wave travelling in one medium strikes upon a second medium having a different dielectric constant, permeability or conductivity, the wave in general will be partially transmitted and partially reflected.

If the expression for the electric field of the incident wave is

$$E_i e^{-j\beta x}$$

and surface of the perfect conductor is taken to be the $x=0$ plane, the expression for reflected wave be,

$$E_r e^{j\beta x}$$

where, E_r must be determined from Boundary condition. This requires that the sum of electric field strength in initial and reflected waves add to give zero resultant field strength in plane $x=0$

$$\text{At } x=0 \quad E_i + E_r = 0$$

$$\boxed{E_i = -E_r}$$

This means, the Amplitude of incident and reflected electric field strengths are equal but with a phase reversal on reflection.

$$\begin{aligned} \text{Now, } E_T(x) &= E_i e^{-j\beta x} + E_r e^{j\beta x} \\ &= E_i (e^{-j\beta x} - e^{j\beta x}) \end{aligned}$$

$$\Rightarrow E_T(x) = -2j E_i \sin \beta x \quad \text{--- ①}$$

$$\hat{E}_T(x, t) = \text{Re} \{ -2j E_i \sin \beta x e^{j\omega t} \}$$

If E_i is chosen to be real

$$\tilde{E}_T(x, t) = 2E_i \sin \beta x \sin \omega t \quad \text{--- (2)}$$

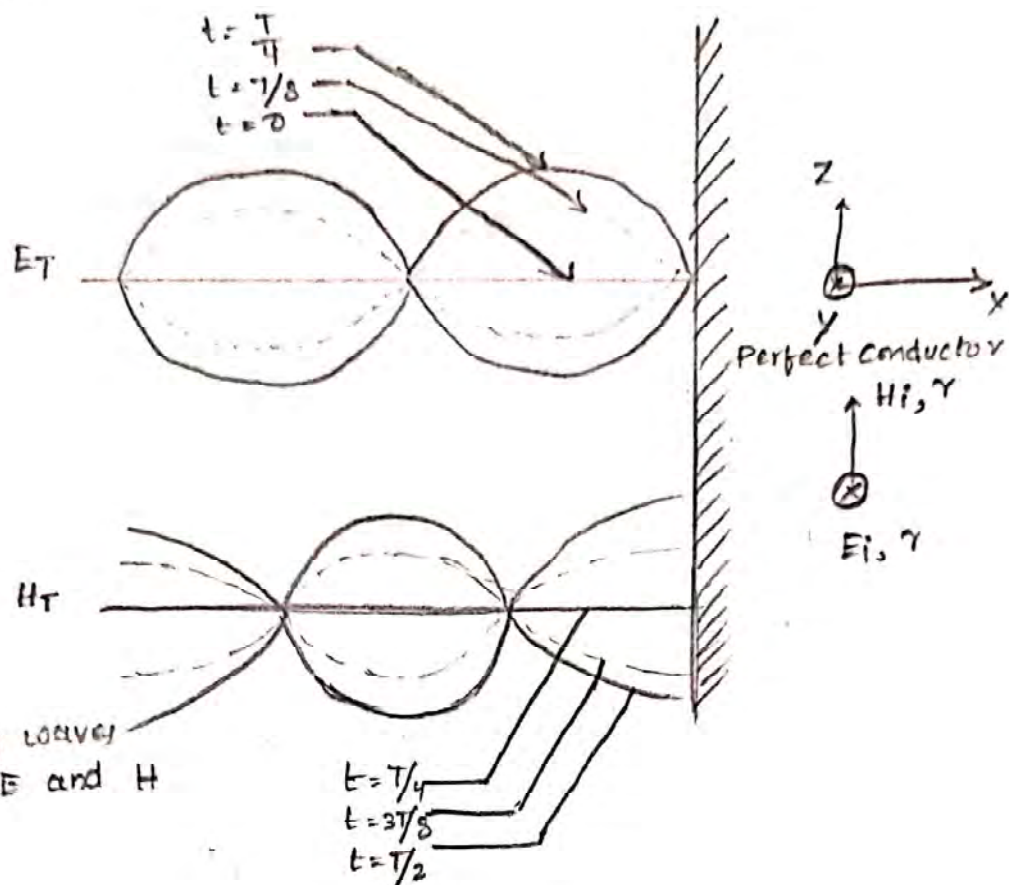


fig: standing waves of E and H

Eq-①. Shows that the incident and reflected waves combine to produce a standing wave, which does not produce.

1. The magnitude of electric wave field varies sinusoidally with distance from the reflecting plane.

2. $E_R = 0$ at surface of conductor ($z=0$) and also at $z = \frac{n\lambda}{2}$, $n=1, 2, 3, \dots$

3. $(E_R)_{\max} = 2E_i$

4. E_R is maximum at $z = m \frac{\lambda}{4}$, $m=1, 3, 5, \dots$

The expression for the resultant magnetic field will be

$$H_T(x) = H_i e^{-j\beta x} + H_r e^{j\beta x}$$

$$= H_i (e^{-j\beta x} + e^{j\beta x}) \rightarrow H_T(x) = 2H_i \cos \beta x \rightarrow \text{--- (3)}$$

H_i is real since it is in phase with E_i

$$\tilde{H}_T(x, t) = \text{Re} \{ H_T(x) e^{j\omega t} \}$$

$$= 2H_i \cos \beta x \cos \omega t \quad \text{--- (4)}$$

Resultant magnetic field strength 'H' also has a standing wave distribution.

$$H_s = H_T \text{ (at } x=0)$$

E_i and H_i were in the time phase in the incident plane wave, a comparison of (1) & (2) shows that E_T and H_T are 90° out of time phase because of the factor j in eq-(1).

That E_T and H_T are 90° apart in time phase can be seen more clearly by rewriting (1) & (3). Replacing $-j$ by its equivalent $e^{-j(\pi/2)}$ and combining this with the $e^{j\omega t}$ term to give $e^{j[\omega t - (\pi/2)]}$ eq-(1) becomes,

$$\tilde{E}_T(x, t) = \text{Re} \{ 2E_i \sin \beta x e^{-j\pi/2} e^{j\omega t} \}$$

$$= 2E_i \sin \beta x \cos(\omega t - \pi/2) \quad \text{--- (5)}$$

likewise rewriting eq (3)

$$H_T = 2H_i \cos \beta x \cos \omega t \quad \text{--- (6)}$$

Comparison of eq-(5) & eq-(6) shows E_T and H_T differ in time phase by $\pi/2$ radians or 90° .

Reflection by a perfect conductor - Oblique Incidence:

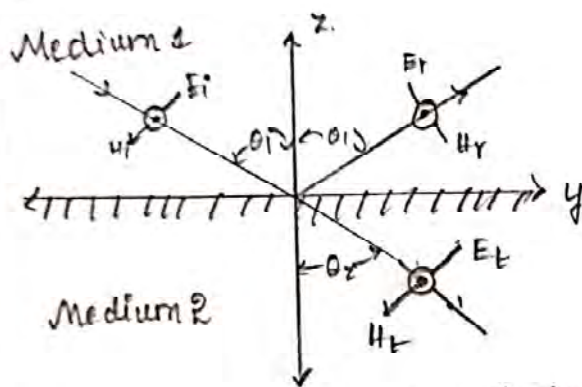
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Whenever a wave is incident obliquely on the interface between two media, it is necessary to consider separately two special cases.

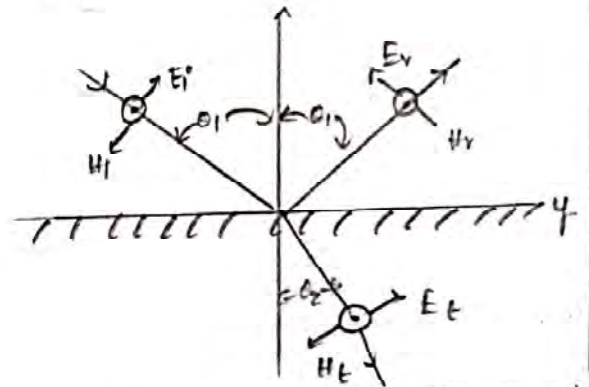
(i). The electric vector is parallel to boundary surface or perpendicular to plane of incidence. This case is often termed "Horizontal polarisation".

(ii). The Magnetic vector is parallel to boundary surface, and the electric vector is parallel to the plane of incidence. This case is often termed "Vertical polarisation".

The two cases are shown in below figure.



(a). perpendicular (Horizontal) polarisation



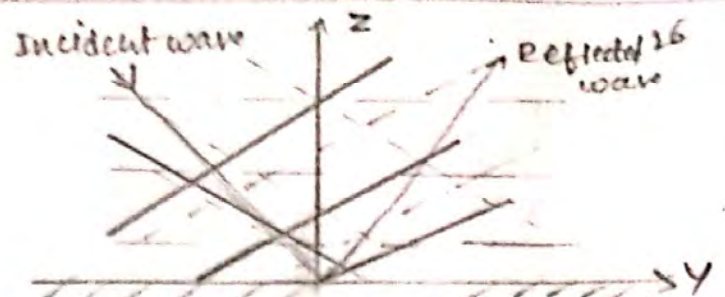
(b). Parallel (vertical) polarisation

The terms 'horizontally' and 'vertically' polarized waves refer to the fact that waves from horizontal and vertical antennas, respectively, would produce these particular orientations of electric and magnetic vectors in waves striking the surface of the earth.

Case - i :-

E perpendicular to the plane of Incidence.

Let the incident and reflected waves make angle $\theta_i = \theta_r = \theta$ with the z -axis as shown in figure below.



Because of these two waves have equal wavelengths and opposite directions along the z -axis, there must be a standing-wave distribution along this axis with the co-ordinate system, chosen as shown in figure, the expression for the reflected wave is

$$E_{\text{reflected}} = E_r e^{-j\beta \hat{n}' \cdot r} \\ = E_r e^{-j\beta (x \cos \theta + y \sin \theta + z \cos \theta)} \quad \text{--- (7)}$$

where E_r is amplitude of electric field strength. For Normal of the reflected wave.

$$\hat{n}' \cdot r = x \cos \frac{\pi}{2} + y \cos \left(\frac{\pi}{2} - \theta \right) + z \cos \theta \\ = y \sin \theta + z \cos \theta$$

so that eq-⑦ becomes

$$E_{\text{reflected}} = E_r e^{-j\beta (y \sin \theta + z \cos \theta)} \quad \text{--- (8)}$$

for incident wave,

$$\hat{n} \cdot r = x \cos \frac{\pi}{2} + y \cos \left(\frac{\pi}{2} - \theta \right) + z \cos (\pi - \theta) \\ = y \sin \theta - z \cos \theta$$

$$\text{and, } E_{\text{incident}} = E_i e^{-j\beta (y \sin \theta - z \cos \theta)} \quad \text{--- (9)}$$

from boundary conditions, $E_r = -E_i$

therefore the total electric field strength will be.

$$E = E_i \left[e^{-j\beta (y \sin \theta - z \cos \theta)} - e^{-j\beta (y \sin \theta + z \cos \theta)} \right] \\ = 2j E_i \sin (\beta z \cos \theta) e^{-j\beta y \sin \theta} \\ = 2j E_i \sin \beta z \times e^{-j\beta y} \quad \text{--- (10)}$$

Where $\beta = \frac{\omega}{v} = \frac{2\pi}{\lambda}$ is the phase-shift constant of the incident wave,

$\beta_x = \beta \cos \theta$ is the phase-shift constant in z-direction

$\beta_y = \beta \sin \theta$ is the phase-shift constant in y-direction.

eq-(10) shows standing wave distribution of electric field strength along the z-axis

The relation between the wavelength is

$$\lambda_z = \frac{2\pi}{\beta_x} = \frac{2\pi}{\beta \cos \theta} = \frac{\lambda}{\cos \theta}$$

The whole standing wave distribution of electric field strength is seen from eq-(10) to be travelling in y-direction with velocity.

$$v_y = \frac{\omega}{\beta_y} = \frac{\omega}{\beta \sin \theta} = \frac{v}{\sin \theta}$$

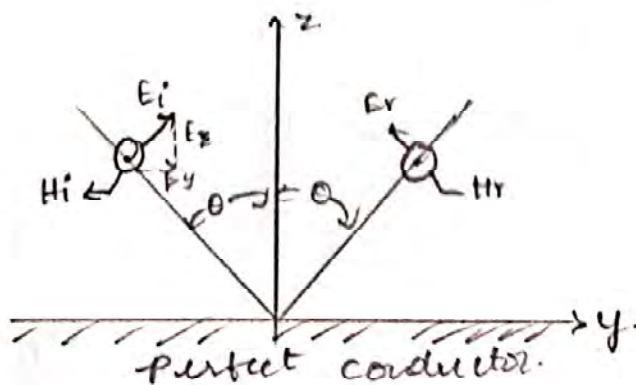
This is the velocity with which a crest of incident wave moves along y-axis.

The wavelength in this direction is $\lambda_y = \frac{\lambda}{\sin \theta}$

Case-II:

E-parallel to plane of Incidence:

In this case E_i and E_r will have instantaneous direction. Because components parallel to the perfectly conducting boundary must be equal and opposite. The magnetic field strength vector H will be reflected without phase reversal as an examination of direction of current flow.



The magnitudes of E and H will be related by

$$\left[\frac{E_i}{H_i} = \frac{E_r}{H_r} = \eta \right]$$

For incident wave the expression for magnetic field strength would be,

$$H_{\text{incident}} = H_i e^{-j\beta y \sin\theta - z \cos\theta}$$

For reflected wave,

$$H_{\text{reflected}} = H_r e^{-j\beta y \sin\theta + z \cos\theta}$$

Since, $H_i = H_r$, the total magnetic field strength is

$$H = 2H_i \cos\beta_z z e^{-j\beta y y} \quad \text{--- (11)}$$

For incident wave,

$$E_i = \eta H_i, \quad E_z = \eta \sin\theta H_i, \quad E_y = \eta \cos\theta H_i$$

For reflected wave

$$E_r = H_r, \quad E_z = \eta \sin\theta H_r, \quad E_y = -\eta \cos\theta H_r$$

The total z component of electric field strength is

$$E_z = 2\eta \sin\theta H_i \cos\beta_z z e^{-j\beta y y} \quad \text{--- (12)}$$

The total y component of electric field strength is

$$E_y = 2j\eta \cos\theta H_i \sin\beta_z z e^{-j\beta y y} \quad \text{--- (13)}$$

Both components of the electric field strength have a standing wave distribution above the reflecting plane.

However for normal or z -component of E the maxima occur at the plane and multiples of $\lambda z/2$ from plane.

Reflection by a Perfect Dielectric - Normal Incidence :

When a plane electromagnetic wave is incident normally on the surface of a perfect dielectric, part of the energy is transmitted and part of it is reflected.

Consider the case of plane wave travelling in the x -direction incident on a boundary that is parallel to the $x=0$ plane.

Let, E_i be electric field strength of incident wave striking the boundary.

E_r be electric field strength of reflected wave leaving the boundary in first medium.

E_t be electric field strength of transmitted wave propagated into second medium.

Let ϵ_1 & μ_1 be constants of first medium; ϵ_2 and μ_2 be constants of second medium.

Designating by η_1 and η_2 the ratios $\sqrt{\frac{\mu_1}{\epsilon_1}}$ & $\sqrt{\frac{\mu_2}{\epsilon_2}}$, the following relations will hold,

$$E_i = \eta_1 H_i$$

$$E_r = -\eta_1 H_r$$

$$E_t = \eta_2 H_t$$

The continuity of the tangential components of E and H require that

$$H_i + H_r = H_t$$

$$E_i + E_r = E_t$$

Combining these,

$$H_i + H_r = \frac{1}{\eta_1} (E_i - E_r) = H_t = \frac{1}{\eta_2} (E_i + E_r)$$

$$\rightarrow \eta_2 (E_i - E_r) = \eta_1 (E_i + E_r)$$

$$E_i (\eta_2 - \eta_1) = E_r (\eta_2 + \eta_1)$$

$$\Rightarrow \frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad \text{--- (14)}$$

Also,

$$\begin{aligned} \frac{E_t}{E_i} &= \frac{E_i + E_r}{E_i} = 1 + \frac{E_r}{E_i} \\ &= \frac{2\eta_2}{\eta_2 + \eta_1} \quad \text{--- (15)} \end{aligned}$$

$$\frac{H_r}{H_i} = -\frac{E_r}{E_i} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} \quad \text{--- (16)}$$

$$\frac{H_t}{H_i} = \frac{\eta_1}{\eta_2} \cdot \frac{E_t}{E_i} = \frac{2\eta_1}{\eta_1 + \eta_2} \quad \text{--- (17)}$$

The permeabilities of all known insulators do not differ appreciably from that of free space, so that

$$\mu_1 = \mu_2 = \mu_v$$

Substituting these relations the above expressions can be written in terms of the dielectric constants as follows.

$$\frac{E_r}{E_i} = \frac{\sqrt{\mu_v/\epsilon_2} - \sqrt{\mu_v/\epsilon_1}}{\sqrt{\mu_v/\epsilon_2} + \sqrt{\mu_v/\epsilon_1}}$$

$$\Rightarrow \frac{E_r}{E_i} = \frac{\sqrt{\epsilon_1} - \sqrt{\epsilon_2}}{\sqrt{\epsilon_1} + \sqrt{\epsilon_2}} \quad \text{--- (18)}$$

Similarly,

$$\frac{E_t}{E_i} = \frac{2\sqrt{\epsilon_1}}{\sqrt{\epsilon_1} + \sqrt{\epsilon_2}} \quad \text{--- (19)}$$

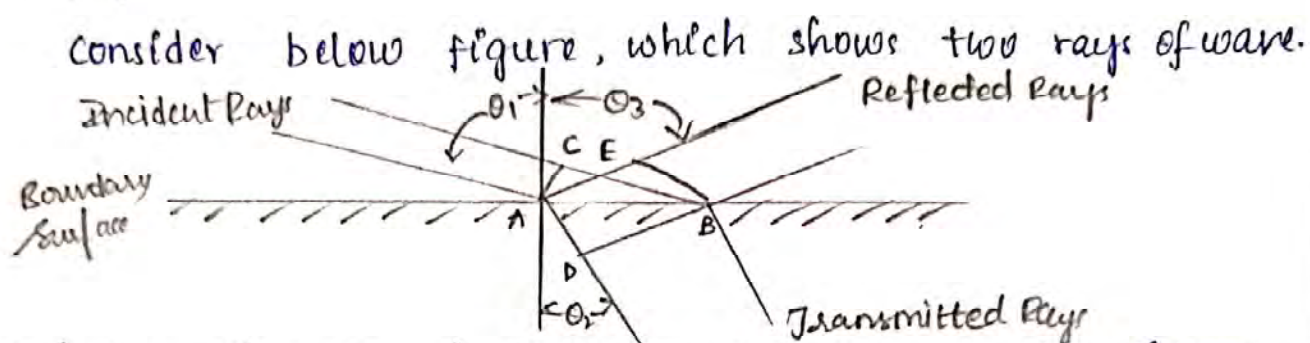
$$\frac{H_r}{H_i} = \frac{\sqrt{\epsilon_2} - \sqrt{\epsilon_1}}{\sqrt{\epsilon_1} + \sqrt{\epsilon_2}} \quad \text{--- (20)}$$

$$\frac{H_t}{H_i} = \frac{2\sqrt{\epsilon_2}}{\sqrt{\epsilon_1} + \sqrt{\epsilon_2}} \quad \text{--- (21)}$$

Reflection by a perfect Insulator - Oblique Incidence :

If a plane wave is incident upon a boundary surface that is not parallel to plane containing $\vec{E}$ and $\vec{H}$.

Again part of wave will be transmitted, and part of it reflected, but in this case the transmitted wave will be refracted that is the direction of propagation will be altered.



In the diagram incident ray (2) travels the distance CB , where as transmitted ray (1) travels the distance AD and reflected ray (1) travels from A to E .

If, v_1 is velocity of the wave in medium (1) and v_2 is the velocity of the wave in medium (2), then

$$\frac{CB}{AD} = \frac{v_1}{v_2}$$

$$\text{Now, } CB = AB \sin \theta_1$$

$$AD = AB \sin \theta$$

$$\text{so that, } \frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}$$

In terms of constants of media v_1 & v_2 are given by

$$v_1 = \frac{1}{\sqrt{\mu_1 \epsilon_1}} = \frac{1}{\sqrt{\mu_v \epsilon_1}}$$

$$v_2 = \frac{1}{\sqrt{\mu_2 \epsilon_2}} = \frac{1}{\sqrt{\mu_v \epsilon_2}}$$

Therefore, $\frac{\sin \theta_1}{\sin \theta_2} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \quad \text{--- (22)}$

Furthermore,

$$AC = CB \quad \text{--- (23)}$$

and as a result, $\sin \theta_1 = \sin \theta_3$

$$\Rightarrow \theta_1 = \theta_3$$

The Angle of incidence is equal to angle of reflection, the angle of incidence is related to angle of refraction by eq-(22), which in Optics is known as the law of sines or Snell's law.

⑧ case -i : perpendicular (horizontal) polarisation:

let electric field strength E_i of incident wave be in the positive x direction, and let the assumed positive directions for E_r and E_t in reflected and transmitted waves also be in positive x -direction.

$$E_i + E_r = E_t$$

$$\frac{E_i}{E_t} = 1 + \frac{E_r}{E_t} \quad \text{--- (24)}$$

Substituting eq-(24) in eq-(23)

$$\frac{E_r^2}{E_i^2} = 1 - \sqrt{\frac{\epsilon_2}{\epsilon_1}} \left(1 + \frac{E_r}{E_i} \right)^2 \frac{\cos \theta_2}{\cos \theta_1}$$

$$1 - \left(\frac{E_r}{E_i} \right)^2 = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \left(1 + \frac{E_r}{E_i} \right)^2 \frac{\cos \theta_2}{\cos \theta_1}$$

$$1 - \frac{E_r}{E_i} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \left(1 + \frac{E_r}{E_i} \right) \frac{\cos \theta_2}{\cos \theta_1}$$

$$\frac{E_r}{E_i} = \frac{\sqrt{\epsilon_1} \cos \theta_1 - \sqrt{\epsilon_2} \cos \theta_2}{\sqrt{\epsilon_1} \cos \theta_1 + \sqrt{\epsilon_2} \cos \theta_2}$$

Now, from Eq. (20),

$$\begin{aligned}\sqrt{\epsilon_2} \cos \theta_2 &= \sqrt{\epsilon_2 (1 - \sin^2 \theta_2)} \\ &= \sqrt{\epsilon_2 - \epsilon_1 \sin^2 \theta_1}\end{aligned}$$

$$\therefore \frac{E_r}{E_i} = \frac{\sqrt{\epsilon_1} \cos \theta_1 - \sqrt{\epsilon_2 - \epsilon_1 \sin^2 \theta_1}}{\sqrt{\epsilon_1} \cos \theta_1 + \sqrt{\epsilon_2 - \epsilon_1 \sin^2 \theta_1}} \quad \text{--- (25)}$$

$$\begin{aligned}&= \frac{\cos \theta_1 - \sqrt{\left(\frac{\epsilon_2}{\epsilon_1}\right) - \sin^2 \theta_1}}{\cos \theta_1 + \sqrt{\left(\frac{\epsilon_2}{\epsilon_1}\right) - \sin^2 \theta_1}} \quad \text{--- (26)}\end{aligned}$$

Eq. (26) gives ratio of reflected to incident electric field strength for case of perpendicularly polarized wave.

Case-II : Parallel (vertical) Polarization :

In this case E is parallel to plane of incidence and H is parallel to reflecting surface.

Applying the boundary conditions that tangential component of E is continuous across boundary in case gives,

$$(E_i - E_r) \cos \theta_1 = E_t \cos \theta_2$$

$$\frac{E_t}{E_i} = \left(1 - \frac{E_r}{E_i}\right) \frac{\cos \theta_1}{\cos \theta_2}$$

substituting this in eq. (23)

$$\left(\frac{E_r}{E_i}\right)^2 = 1 - \sqrt{\frac{\epsilon_2}{\epsilon_1}} \left(1 - \frac{E_r}{E_i}\right)^2 \frac{\cos \theta_1}{\cos \theta_2}$$

$$1 - \frac{E_r^2}{E_i^2} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \left(1 - \frac{E_r}{E_i}\right)^2 \frac{\cos \theta_1}{\cos \theta_2}$$

$$1 + \frac{E_r}{E_i} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \left(1 - \frac{E_r}{E_i}\right) \frac{\cos \theta_1}{\cos \theta_2}$$

$$\frac{E_r}{E_i} \left(1 + \sqrt{\frac{\epsilon_1}{\epsilon_2}} \frac{\cos \theta_1}{\cos \theta_2}\right) = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \frac{\cos \theta_1}{\cos \theta_2} - 1$$

$$\frac{E_r}{E_i} = \frac{\sqrt{\epsilon_2} \cos \theta_1 - \sqrt{\epsilon_1} \cos \theta_2}{\sqrt{\epsilon_2} \cos \theta_1 + \sqrt{\epsilon_1} \cos \theta_2}$$

$$= \frac{\sqrt{\epsilon_2} \cos \theta_1 - \sqrt{\epsilon_1 (1 - \sin^2 \theta_2)}}{\sqrt{\epsilon_2} \cos \theta_1 + \sqrt{\epsilon_1 (1 - \sin^2 \theta_2)}}$$

WKT, $\sin^2 \theta_2 = \frac{\epsilon_1}{\epsilon_2} \sin^2 \theta_1$

$$\frac{E_r}{E_i} = \frac{\left(\frac{\epsilon_2}{\epsilon_1}\right) \cos \theta_1 - \sqrt{\left(\frac{\epsilon_2}{\epsilon_1}\right) - \sin^2 \theta_1}}{\left(\frac{\epsilon_2}{\epsilon_1}\right) \cos \theta_1 + \sqrt{\left(\frac{\epsilon_2}{\epsilon_1}\right) - \sin^2 \theta_1}} \quad \text{--- (27)}$$

Eq. (27), gives the reflection coefficient for parallel or vertical polarisation, that is the ratio of reflected to incident field strength when E is parallel to the plane of incidence.

⊛

Brewster Angle:

Brewster Angle is an angle at which no reflection takes place. This occurs when numerator of eq. (27) is zero.

$$\frac{E_r}{E_i} = \frac{\left(\frac{\epsilon_2}{\epsilon_1}\right) \cos \theta_1 - \sqrt{\left(\frac{\epsilon_2}{\epsilon_1}\right) - \sin^2 \theta_1}}{\left(\frac{\epsilon_2}{\epsilon_1}\right) \cos \theta_1 + \sqrt{\left(\frac{\epsilon_2}{\epsilon_1}\right) - \sin^2 \theta_1}} \quad \text{--- (27)}$$

$$\frac{\epsilon_2}{\epsilon_1} \cos \theta_1 - \sqrt{\left(\frac{\epsilon_2}{\epsilon_1}\right) - \sin^2 \theta_1} = 0$$

$$\Rightarrow \frac{\epsilon_2}{\epsilon_1} \cos \theta_1 = \sqrt{\left(\frac{\epsilon_2}{\epsilon_1}\right) - \sin^2 \theta_1}$$

$$\Rightarrow \frac{\epsilon_2^2}{\epsilon_1^2} \cos^2 \theta_1 = \frac{\epsilon_2}{\epsilon_1} - \sin^2 \theta_1$$

$$\frac{\epsilon_2^2}{\epsilon_1^2} (1 - \sin^2 \theta_1) = \frac{\epsilon_2}{\epsilon_1} - \sin^2 \theta_1$$

$$\frac{\epsilon_2^2}{\epsilon_1^2} - \frac{\epsilon_2^2}{\epsilon_1^2} \sin^2 \theta_1 = \frac{\epsilon_2}{\epsilon_1} - \sin^2 \theta_1$$

$$\rightarrow \sin^2 \theta_1 - \frac{e_2^2}{e_1^2} \sin^2 \theta_1 = \frac{e_2}{e_1} - \frac{e_2^2}{e_1^2}$$

$$(e_1^2 - e_2^2) \sin^2 \theta_1 = e_2 (e_1 - e_2)$$

$$\rightarrow \sin^2 \theta_1 = \frac{e_2 (e_1 - e_2)}{e_1^2 - e_2^2}$$

$$\sin^2 \theta_1 = \frac{e_2}{e_1 + e_2}, \quad \cos^2 \theta_1 = \frac{e_1}{e_1 + e_2}, \quad \tan^2 \theta_1 = \frac{e_2}{e_1}$$

$$\Rightarrow \tan \theta = \sqrt{\frac{e_2}{e_1}}$$

$$\theta_1 = \tan^{-1} \sqrt{\frac{e_2}{e_1}}$$

As, this angle which is called Brewster angle, there is no reflected wave when the incident wave is parallel polarized.

③ TOTAL - INTERNAL REFLECTION :

When a wave is incident from the denser medium into rarer medium at an angle equal to or greater than the critical angle the wave will be totally internally reflected back. This phenomenon is called total internal reflection.

If e_1 is greater than e_2 both the reflection coefficients for vertical and horizontal polarisations become complex when,

$$\sin \theta_1 > \sqrt{\frac{e_2}{e_1}}$$

At critical angle the reflection coefficients has unity value.

$$\frac{E_r}{E_i} = \frac{\cos \theta_1 - \sqrt{\frac{e_2}{e_1} - \sin^2 \theta_1}}{\cos \theta_1 + \sqrt{\frac{e_2}{e_1} - \sin^2 \theta_1}} = 1$$

But $\theta_1 = \theta_c$ (critical angle)

$$\cos \theta_c - \sqrt{\frac{e_2}{e_1} - \sin^2 \theta_c} = \cos \theta_c + \sqrt{\frac{e_2}{e_1} - \sin^2 \theta_c}$$

$$\Rightarrow 2\sqrt{\frac{\epsilon_2}{\epsilon_1}} \cdot \sin^2 \theta_c = 0$$

$$\frac{\epsilon_2}{\epsilon_1} \cdot \sin^2 \theta_c = 0$$

$$\sin^2 \theta_c = \frac{\epsilon_2}{\epsilon_1}$$

$$\boxed{\theta_c = \sin^{-1} \sqrt{\frac{\epsilon_2}{\epsilon_1}}}$$

This is the critical angle at which there is no refracted wave. It is widely used in binocular optics where glass prisms are employed to shorten instrument, thus making it easier to carry.



Surface Impedance :-

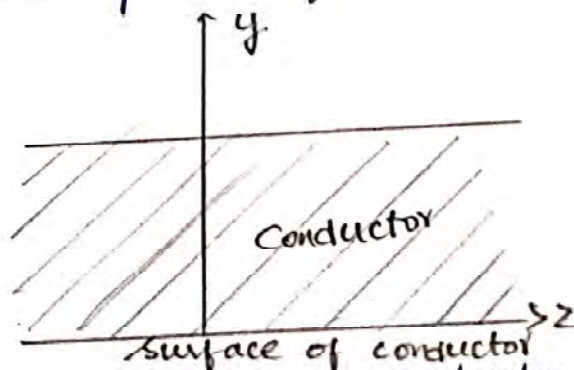
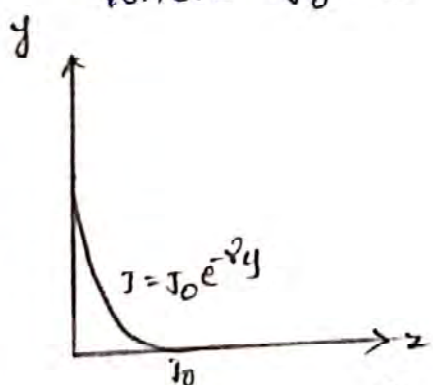
It is defined as the ratio of the tangential component of electric field at surface of conductor to linear current density.

$$Z_s = \frac{E_{tan}}{J_s}$$

Consider a flat plate conductor with its surface at $y=0$ plane, the current distribution in y direction will be given by,

$$J = J_0 e^{-\gamma y}$$

where J_0 is current density at surface.



It is assumed that the thickness of conductor is very much greater than depth of penetration, so that there is no reflection from the back surface of the conductor.

The total conduction current per meter width is —
 linear current density i_s ,

$$I_s = \int_0^{\infty} J dy = J_0 \int_0^{\infty} e^{-\delta y} dy$$

$$= -\frac{J_0}{\delta} [e^{-\delta y}]_0^{\infty}$$

$$\Rightarrow I_s = \frac{J_0}{\delta}$$

But J_0 , the current density at surface is,

$$J_0 = \sigma E \tan \delta$$

$$\therefore Z_s = \frac{E \tan \delta}{I_s}$$

$$= \frac{E \tan \delta}{J_0 / \delta}$$

$$= \frac{E \tan \delta}{\sigma E \tan \delta}$$

$$\Rightarrow Z_s = \frac{\delta}{\sigma}$$

The constant δ for propagation in a conducting medium is,

$$\delta = \sqrt{\frac{2}{f \omega \mu (\sigma + j \omega \epsilon)}} \approx \sqrt{\frac{2}{f \omega \mu \sigma}}$$

$$\text{So, } Z_s = \frac{\delta}{\sigma}$$

$$= \sqrt{\frac{2}{f \omega \mu \sigma}}$$

$$= \sqrt{\frac{2}{f \omega \mu}} = \eta \quad (\text{for conducting medium})$$

For Good conductors the surface Impedance of a plane conductor that is very much thicker than the skin depth is just Equal to characteristic impedance of the conductor.

Under these conditions, the I/p Impedance is approximately Equal to the I/p Impedance of a lossy line terminated in an open circuit viz,

$$Z_{in} = \eta \coth \gamma l$$

where, l is the thickness of conductor

η is intrinsic impedance.

γ is propagation constant

For good conductor $\sigma \gg \omega \epsilon$ and if conductor thickness is very much greater than depth of penetration, the surface impedance of such a conductor is,

$$Z_s = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ$$

The surface resistance is

$$R_s \approx \sqrt{\frac{\omega\mu}{2\sigma}}$$

and surface reactance has same magnitude as R_s at all frequencies, $X_s \approx \sqrt{\frac{\omega\mu}{2\sigma}}$

The expression for depth of penetration in conductor is

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

it is seen that, $R_s = \frac{1}{\sigma\delta}$

The surface resistance of a flat conductor at any frequency is Equal to the DC Resistance of thickness of same conductor, where δ is depth of penetration/skin depth.

Poynting Theorem :-

As electromagnetic waves propagate through space from their source to distant receiving paths, there is a transfer of energy from the source to receivers.

There exists a simple and direct relation between the rate of this energy transfer and the amplitudes of Electric & Magnetic field strengths of electromagnetic wave.

This relation can be obtained from Maxwell's Equation as follows :

The magnetomotive force equation I is

$$\nabla \times H = J + \epsilon \frac{\partial E}{\partial t}$$
$$\Rightarrow \boxed{J = \nabla \times H - \epsilon \frac{\partial E}{\partial t}} \quad \text{--- (I)}$$

This Expresses a relation between quantities which have the dimensions of current density.

If it is multiplied by 'E' there will be result a relation between quantities which will have the dimensions of power per unit volume,

$$\text{i.e.} \quad E \cdot J = E \cdot \nabla \times H - \epsilon E \cdot \frac{\partial E}{\partial t} \quad \text{--- (1)}$$

From, Vector Analysis,

$$\nabla \cdot (E \times H) = H \cdot \nabla \times E - E \cdot \nabla \times H$$

$$\Rightarrow E \cdot \nabla \times H = H \cdot \nabla \times E - \nabla \cdot (E \times H) \quad \text{--- (2)}$$

Substituting eq. (2) in eq. (1)

$$E \cdot J = H \cdot \nabla \times E - \nabla \cdot (E \times H) - \epsilon E \cdot \frac{\partial E}{\partial t} \quad \text{--- (3)}$$

Introducing second field equation

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$$\nabla \times E = -\mu \frac{\partial H}{\partial t} \quad \text{--- (II)}$$

Obtain

$$E \cdot J = -\mu H \cdot \frac{\partial H}{\partial t} - \epsilon E \cdot \frac{\partial E}{\partial t} - \nabla \cdot E \times H \quad \text{--- (4)}$$

Now,

$$H \cdot \frac{\partial H}{\partial t} = \frac{1}{2} \frac{\partial H^2}{\partial t}$$

$$E \cdot \frac{\partial E}{\partial t} = \frac{1}{2} \frac{\partial E^2}{\partial t}$$

So that,

$$E \cdot J = -\frac{\mu}{2} \frac{\partial}{\partial t} H^2 - \frac{\epsilon}{2} \frac{\partial}{\partial t} E^2 - \nabla \cdot E \times H$$

Integrating over a volume V ,

$$\int_V E \cdot J dv = -\frac{\partial}{\partial t} \int_V \left(\frac{\mu}{2} H^2 + \frac{\epsilon}{2} E^2 \right) dv - \int_V \nabla \cdot E \times H dv \quad \text{--- (5)}$$

using divergence theorem the last term can be changed from a volume integral to a surface integral over the surrounding surface S .

$$\text{i.e.} \quad \int_V \nabla \cdot E \times H dv = \oint_S E \times H \cdot da$$

Then eq. (5) can be written as

$$\int_V E \cdot J dv = -\frac{\partial}{\partial t} \int_V \left(\frac{\mu}{2} H^2 + \frac{\epsilon}{2} E^2 \right) dv - \oint_S E \times H \cdot da \quad \text{--- (6)}$$

The term on the left hand side represents power dissipated in the volume V .

This result is obtained as a generalisation of Joule's law.

A conductor of cross sectional area ' A ', carrying a current I and having a voltage drop ' E ' per unit length will have a power loss of EI watts per unit length.

The power dissipated per unit volume would be.

$$\frac{EI}{A} = EJ \text{ watts per unit volume}$$

In this case E and J are in same direction

The power dissipated per unit volume would always be given by
" $E \cdot J$ "

and total power dissipated in a volume v , would be

$$\int_V E \cdot J dv \quad \text{--- (7)}$$

The first term in eq. (6), the quantity $\frac{1}{2} \epsilon E^2$ could be considered to represent energy density or stored electric energy per unit volume of electric field.

$\frac{1}{2} \mu H^2 \rightarrow$ represents stored energy density of magnetic field.

The term, $-\oint_S E \times H \cdot da$ represents the rate of flow of energy inward through the surface of the volume.

Then, this expression without '-ve' sign $\oint_S E \times H \cdot da$ which represents rate of flow of energy outward through the surface enclosing the volume.

It is seen that vector $\boxed{P = E \times H}$, has the dimensions of watts per square meter.

⊗

Poynting vector :

The poynting vector (P) measures the rate of flow of energy of the wave as it propagates.

⊗ Instantaneous, Average and Complex Poynting vector :

In an ac circuit, the instantaneous power $\tilde{w}$ is always given by the product of instantaneous voltage $\tilde{v}$ and instantaneous current $\tilde{i}$.

$$\tilde{w} = \tilde{v} \tilde{i}$$

In the equation, the quantities $\tilde{v}$ and $\tilde{i}$ may be represented in terms of phasors v and i as follows

$$\begin{aligned}\tilde{v} &= \text{Re}\{V e^{j\omega t}\} \\ &= \text{Re}\{|V| e^{j\theta_v} e^{j\omega t}\}\end{aligned}$$

$$\Rightarrow \tilde{v} = |V| \cos(\omega t + \theta_v) \quad \text{--- (1)}$$

$$\begin{aligned}\tilde{i} &= \text{Re}\{I e^{j\omega t}\} \\ &= \text{Re}\{|I| e^{j\theta_i} e^{j\omega t}\}\end{aligned}$$

$$\Rightarrow \tilde{i} = |I| \cos(\omega t + \theta_i) \quad \text{--- (2)}$$

Now the "instantaneous power" may be expressed as

$$\tilde{w} = |V| |I| \cos(\omega t + \theta_v) \cos(\omega t + \theta_i)$$

$$\tilde{w} = \frac{|V| |I|}{2} [\cos(\theta_v - \theta_i) + \cos(2\omega t + \theta_v + \theta_i)] \quad \text{--- (3)}$$

which consists of average part & an oscillating part

From figure,

$$\theta_v - \theta_i = \theta$$

$$\text{we may write } w_{av} = \frac{|V| |I|}{2} \cos \theta \quad \text{--- (4)}$$

Another quantity is "Reactive power"

or more accurately, the reactive volt amperes (VA)

$$w_{react} = \frac{|V| |I|}{2} \sin \theta \quad \text{--- (5)}$$

Since θ is the phase angle between voltage and current w_{av} and w_{react} are the in-phase and out-of-phase components of volt-ampere product.

The 'complex power' W is defined as one-half the product of V and complex conjugate of I .

$$\begin{aligned} W &= \frac{1}{2} V I^* \\ &= \frac{1}{2} |V| e^{j0} |I| e^{-j\theta} \\ &= \frac{|V| |I|}{2} e^{j0} \end{aligned}$$

$$\Rightarrow W = W_{av} + jW_{react} \quad \text{--- (6)}$$

This shows that average and reactive power may be recovered from complex power by taking real and imaginary parts

$$W_{av} = \frac{1}{2} \operatorname{Re} \{ V I^* \}$$

$$W_{react} = \frac{1}{2} \operatorname{Im} \{ V I^* \}$$

In electromagnetic field theory there are relations similar to above between Poynting vector.

The instantaneous power flow per square meter is

$$\vec{P} = \vec{E} \times \vec{H} \quad \text{--- (7)}$$

The complex Poynting vector P is defined as

$$P = \frac{1}{2} E \times H^* \quad \text{--- (8)}$$

from which we may obtain the average and reactive parts of power per square meter.

$$P_{av} = \frac{1}{2} \operatorname{Re} \{ E \times H^* \}$$

$$P_{react} = \frac{1}{2} \operatorname{Im} \{ E \times H^* \}$$

Thus in rectangular co-ordinates, the complex flow of power per unit area normal to y - z plane is

$$P_x = \frac{1}{2} (E_y H_z^* - E_z H_y^*) \quad \text{--- (9)}$$

In spherical co-ordinate, the outward flow of complex

$$\text{power per unit area } P_r = \frac{1}{2} (E_\theta H_\phi^* - E_\phi H_\theta^*) \quad \text{--- (10)}$$

Power loss in a plane conductor :

An Evaluation of the normal component of Poynting vector at the surface of a conductor will give the power flow per unit area through the surface and hence the power loss in the conductor.

Let there be a tangential component of magnetic field strength H_{tan} at the surface of a metallic conductor.

From the continuity Requirements across the boundary surface the tangential component of H just inside the conductor will have this same value H_{tan} .

Inside the conductor E_{tan} , the tangential component of E is related to H_{tan} by.

$$\frac{E_{tan}}{H_{tan}} = \eta_m$$

where η_m is the intrinsic impedance of conductor.

$$\begin{aligned} \text{and, } \eta_m &= \sqrt{\frac{j\omega\mu_m}{\sigma_m + j\omega\epsilon_m}} \\ &\approx \sqrt{\frac{j\omega\mu_m}{\sigma_m}} \\ &= \sqrt{\frac{\omega\mu_m}{\sigma_m}} \angle 45^\circ \end{aligned}$$

The average power flow per unit area normal to the surface will be,

$$P_o(\text{real}) = \frac{1}{2} \text{Re} (I_{tan} \times H_{tan}^*) \quad \text{--- (1)}$$

When E_{tan} & H_{tan} are at right angles, and since for any good conductor E_{tan} leads H_{tan} by 45° in time phase, eq. (1) becomes

$$P_n = \frac{1}{2} |E_{tan}| |H_{tan}| \cos 45^\circ$$

$$= \frac{1}{2\sqrt{2}} |\eta_m| |H_{tan}|^2$$

$$P_n = \frac{1}{2\sqrt{2}} \frac{|E_{tan}|^2}{|\eta_m|} \quad \text{--- (2)}$$

For a conductor which has a thickness very much greater than skin depth δ , the surface impedance Z_s is equal to intrinsic impedance η_m of conductor, so that

$$P_n = \frac{1}{2\sqrt{2}} |Z_s| |H_{tan}|^2$$

$$= \frac{1}{2\sqrt{2}} \frac{|E_{tan}|^2}{|Z_s|} \text{ watt/sqm} \quad \text{--- (3)}$$

In a conductor the linear current density J_s is equal in magnitude to the tangential magnetic field strength at surface, $P_n = \frac{1}{2\sqrt{2}} |Z_s| |J_s|^2 \text{ watt/sqm} \quad \text{--- (4)}$

In expressions (3) & (4) E_{tan} , H_{tan} and J_s are peak values. In terms of effective values,

$$P_n = \frac{1}{\sqrt{2}} \frac{|E_{t(eff)}|^2}{|Z_s|}$$

$$= \frac{1}{\sqrt{2}} |Z_s| |H_{t(eff)}|^2$$

$$= \frac{1}{\sqrt{2}} |Z_s| |J_{s(eff)}|^2$$

$$= P_s J_s^2 (eff) \text{ watt/sqm} \quad \text{--- (5)}$$

UNIT-4*TRANSMISSION LINES-1*

TOPICS OF TRANSMISSION LINES - 1 :-

1. Transmission Lines
2. Types of Transmission Lines
3. Applications of Transmission Lines
4. Equivalent circuit of a pair of Transmission Lines
5. primary constants
6. Transmission line Equations
7. Input Impedance of a Transmission Line.
8. Secondary Constants
9. Lossless Transmission Lines
10. Distortion Less Line
11. Phase and Group Velocities
12. Loading OF Lines
13. Input impedance of lossless Transmission Line.

Transmission Lines:-

A Transmission line means of transfer of information from one point to another. Usually it consists of two conductors. It is used to connect a source to a load. The source may be a transmitter and the load may be a receiver.

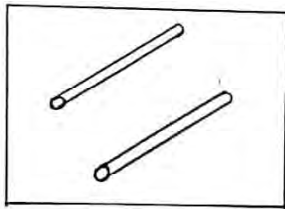
Types OF Transmission Lines:-

The various types of transmission lines are

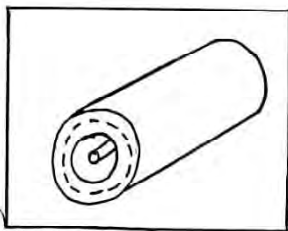
1. Two-wire parallel lines
2. Co-axial lines
3. Twisted wires
4. Parallel plates (or) planar lines
5. Wire above conducting line
6. Microstrip line.
7. Optical fibers

* Typical configurations of above are shown :

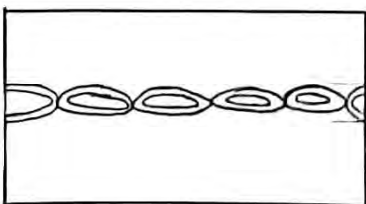
a) Two-wire parallel line



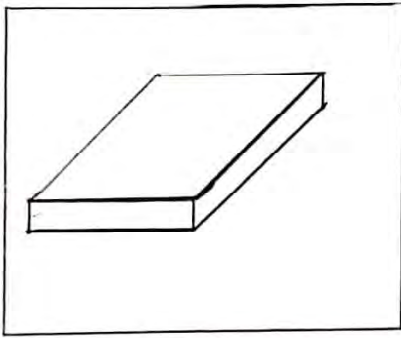
b) Co-axial lines



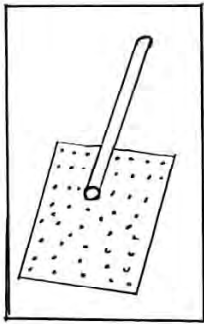
c) Twisted pair of lines



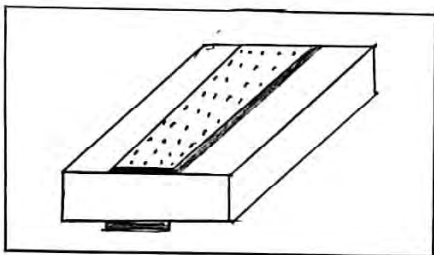
d) planar lines



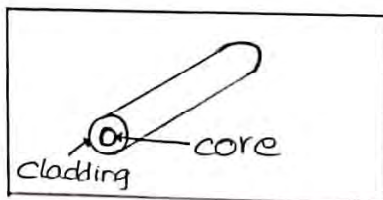
e) wire above conducting plane



f) Microstrip line



g) A typical step-index optical fibers



Application's of Transmission Lines:-

Application's of Transmission lines are:

1. They are used transfer energy from one circuit to another.
2. They can be used as circuit elements like inductors, capacitors and so on..
3. They can be used as impedance matching devices.
4. They can be used as studs.
5. They can be used as measuring devices.
6. Coaxial cables are frequently used in laboratories and to connect televisions to TV antennas
7. Microstrips are used in integrated circuits in which metallic strips connecting electronic elements are deposited on dielectric substrates
8. Twisted pairs and co-axial cables are used in computer networks such as ethernet and Internet.
9. pair of parallel lines are used in telephony and power transmission.
10. planar lines are used to connect transmitters and antennas
11. Optical fibers are used to transmit information over long and short distances with negligible attenuation

Equivalent Circuit of a pair of Transmission Lines:

The equivalent circuit of a transmission line is a "distributed network".

This consists of cascade sections and each section consists of a series Resistance ' R ', series inductance ' L ', capacitance ' C ',

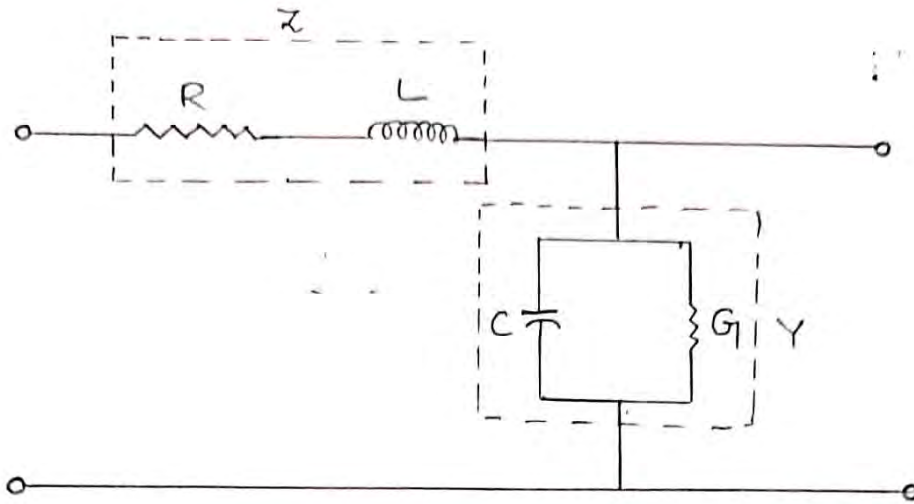
here R is expressed in Ohm's/unit (Ω)

C is expressed in farad/unit length (F)

L is expressed in Henry/unit length (H)

G is expressed in Mho/per unit length (Ω)

Equivalent circuit of a two-conductor transmission line.

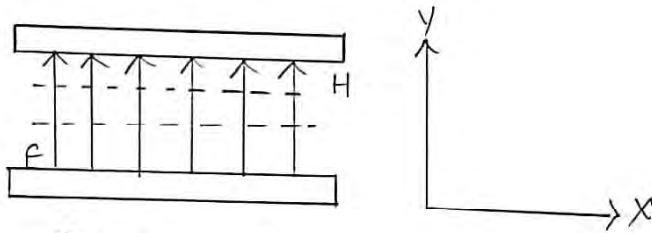


$$Z = R + j\omega L$$

$$Y = G + j\omega C$$

Electric And Magnetic Fields in parallel plate and co-axial Lines:-

In parallel plate transmission, if z is the direction of propagation, electric and magnetic fields are the distributions.



Primary Constants:-

The $R(\Omega/\text{km})$, $L(\text{H}/\text{km})$, $C(\text{F}/\text{km})$ and $G(\text{mho}/\text{km})$ are known as primary constants.

Sailent aspects of primary constants:

1. R is defined as loop resistance per unit length of line.
2. L is defined as Loop inductance Per unit length of line.
3. C is defined as shunt capacitance between two wires per unit length.
4. G is defined as Conductance per unit length due to dielectric medium separating the conductors.

$$[G \neq \frac{1}{R} \div x]$$

5. R, L, C, G are distributed along the length of line.

For each line, the conductors are characterised by σ_c and $\mu_c = \mu_0$, $\epsilon_c = \epsilon_0$, and the dielectric medium, which is basically homogeneous. Separating the conductors is characterised by σ_d , μ_d , ϵ_d .

* for parallel wires,

$$R_{dc} = \frac{2}{\sigma_c \pi a^2}, \Omega/m \text{ for } f < 10 \text{ KHz.}$$

$$R_{ac} = \frac{1}{\pi \sigma_c \delta a}, \Omega/m \text{ for } f > 10 \text{ KHz}$$

$$L = \left[\mu_r + 9.21 \log_{10} \frac{d}{a} \right] \times 10^{-7} \text{ H/m}$$

$$C = \frac{\pi \epsilon_d}{\ln\left(\frac{d}{a}\right)} \text{ F/m}$$

$$G = \frac{C}{\epsilon_d} \sigma_d \text{ mho/m.}$$

Where,

σ_c = conductivity of conductors

a = radius of wire

$\delta = \frac{2}{\sqrt{\pi f \mu_c \sigma_c}} = \text{skin depth}$

d = spacing between wires

ϵ_d = dielectric constant of dielectric material

μ_r = relative permeability of conductor material
= 1 for non-magnetic material

* Internal inductance (L_i):-

It is due to internal flux linkages in the conductors

$$L_i = \frac{R_{ac}}{2\pi f} \text{ (H/m) for } f > 10 \text{ KHz.}$$

$$= \frac{\mu_0}{4\pi} \text{ (H/m) for } f < 10 \text{ KHz}$$

* External Inductance (L_e):-

It is due to flux linkage with the flux external to the wire.

$$L_e = \frac{\mu_d}{\pi} \times \frac{1}{\ln\left(\frac{d}{a}\right)} \text{ H/m for } d \gg a$$

For co-axial cable,

$$R_{dc} = \frac{1}{\sigma_c \pi} \times \left[\frac{1}{a^2} + \frac{1}{t(b+t)} \right]$$

$$R_{ac} = \frac{1}{2\pi \sigma_c \delta} \left[\frac{1}{a} + \frac{1}{b} \right] \text{ for } t \gg \delta$$

$$L_i = \frac{R_{ac}}{2\pi f} \text{ (H/m) for } f > 10 \text{ kHz}$$

$$= \frac{\mu_0}{4\pi} \text{ (H/m) for } f < 10 \text{ kHz}$$

$$L = L_i + L_e.$$

$$C = \pi \epsilon_d \left[\frac{2}{\ln\left(\frac{b}{a}\right)} \right], \text{ (F/m)}$$

$$G = \frac{C}{\epsilon_d} \sigma_d \text{ (mho/m)}$$

where,

a = inner radius

b = outer radius

t = outer thickness.

For parallel plates,

$$R_{dc} = \frac{2}{\sigma_c w t}$$

$$R_{ac} = \frac{2}{\pi \sigma_c \delta w} \text{ for } t \gg \delta$$

$$L_i = \frac{R_{ac}}{2\pi f}, \text{ (H/m) for } f > 10 \text{ kHz}$$

$$= \frac{\mu_0}{4\pi} \text{ (H/m) for } f < 10 \text{ kHz}$$

$$L_e = \frac{\mu_d d}{w} \text{ (H/m)}$$

$$C = \frac{\pi^2 \epsilon_d d}{w} \text{ (F/m)}$$

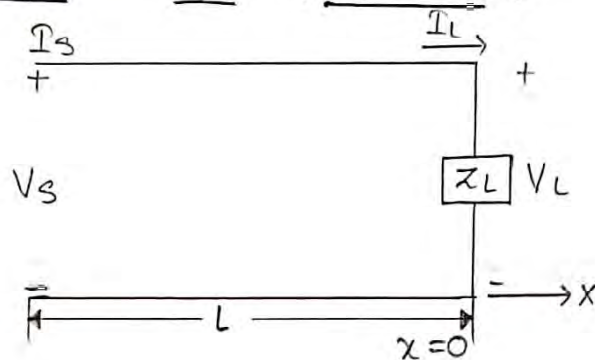
$$G = \frac{C}{\epsilon_d} \sigma_d \text{ (mho/m)}$$

where

w = Width

d = separation

Transmission Line Equation's:-



consider from the above figure in for which a line of length L , voltage and current at source end, ' V_s ' and ' I_s ' are shown. The voltage ' V_L ' and current ' I_L ' at the load end are also shown.

let V and I be voltage and current on line at any arbitrary location, Assume V_f and I_f are for forward wave and V_r and I_r be the reflected wave.

Consider

$$Z = R + j\omega L$$

$Y = G + j\omega C$, from equivalent circuit of a two conductor transmission line.

$$\frac{dV}{dx} = -(R + j\omega L) I$$

$$\frac{dI}{dx} = -(G + j\omega C) V$$

Differentiating and combining, we get.

$$\frac{d^2 V}{dx^2} = \gamma^2 V \text{ ————— } \textcircled{1}$$

$$\frac{d^2 I}{dx^2} = \gamma^2 I \text{ ————— } \textcircled{2}$$

where, $Y^2 = (R + j\omega L)(G + j\omega C)$

$$Y = \sqrt{(R + j\omega L)(G + j\omega C)}$$

where γ = propagation constant.

And the series Impedance

$$Z = (R + j\omega L)$$

And for shunt admittance

$$Y = (G + j\omega C)$$

where

$$\gamma = \sqrt{ZY}$$

From the equation's ① and ② these expressions are in either exponential (or) in hyperbolic forms.

$$V = V_f e^{-\gamma x} + V_r e^{\gamma x}$$

$$I = I_f e^{-\gamma x} + I_r e^{\gamma x}$$

These represent's sum of the forward and reflected waves,

$$Z_0 = \frac{V_f}{I_f} \text{ (or) } - \frac{V_r}{I_r}$$

Z_0 is related to R, L, C and G and is given by,

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{(R + j\omega L)}{(G + j\omega C)}}$$

If Line is terminated in a load impedance of Z_L , then

$$Z_L = \frac{V}{I} = \frac{V_f + V_r}{I_f + I_r}$$

$$= \frac{V_f + V_r}{\frac{V_f}{Z_0} + - \frac{V_r}{Z_0}}$$

$$= Z_0 \frac{(V_f + V_r)}{(V_f - V_r)}$$

(OR)

$$Z_L V_f - Z_L V_r = Z_0 V_f + Z_0 V_r.$$

Or,

Dividing both sides by V_f , we get.

$$Z_L - Z_0 \frac{V_r}{V_f} = Z_0 + Z_0 \frac{V_r}{V_f}$$

Or,

$$\frac{V_r}{V_f} [Z_L + Z_0] = Z_L - Z_0$$

Or,

$$\boxed{\frac{V_r}{V_f} = \frac{Z_L - Z_0}{Z_L + Z_0}}$$

The reflection co-efficient,

$$\rho = \frac{V_r}{V_f} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Input Impedance of a Transmission Line:-

Input Impedance z_i is defined as the ratio of voltage and current at the sending end.

In hyperbolic form, the equations ① and ② are written as,

$$V_1 = A_1 \cosh \gamma x + B_1 \sinh \gamma x$$

$$I_1 = A_2 \cosh \gamma x + B_2 \sinh \gamma x$$

The constants A_1, A_2, B_1 and B_2 are found using the boundary conditions,

$$V = V_L; I = I_L \text{ at } x=0$$

$$V = V_S; I = I_S \text{ at } x=l$$

Then,

$$A_1 = V_L, B_1 = -\sqrt{\frac{R + j\omega L}{G + j\omega C}} I_L$$

$$A_2 = I_L, B_2 = -\sqrt{\frac{G + j\omega C}{R + j\omega L}} V_L$$

$$V_S = V_L \cosh \gamma x - Z_0 I_L \sinh \gamma x$$

$$I_S = I_L \cosh \gamma x + \frac{V_L}{Z_0} \sinh \gamma x$$

z_L is chosen at $x=0$ and if $l=-x$, we have

$$V_s = V_L \cosh \gamma l + z_0 I_L \sinh \gamma l \longrightarrow (3)$$

$$I_s = I_L \cosh \gamma l + \frac{V_L}{z_0} \sinh \gamma l \longrightarrow (4)$$

Here, l is measured from the load end. From equation's 3 and 4 we have

$$z_i = \frac{V_s}{I_s} = \frac{V_L \cosh \gamma l + z_0 I_L \sinh \gamma l}{I_L \cosh \gamma l + \left(\frac{V_L}{z_0}\right) \sinh \gamma l}$$

If the line is short circuited at the receiving end, $z_L=0$ and $V_L=0$. Then the input impedance is

$$(z_i)_{sc} = z_0 \tanh \gamma l$$

Now, if $z_L=\infty$, $I_L=0$, then input impedance is

$$(z_i)_{oc} = z_0 \coth \gamma l$$

OR,

$$(z_i)_{sc}(z_i)_{oc} = z_0^2$$

Secondary Constants:-

The secondary constants are

1. Propagation constant, γ and
2. characteristic impedance, z_0 .

Propagation Constant.

* 1. $\gamma = \sqrt{\text{series impedance} \times \text{shunt admittance}}$

$$= \sqrt{ZY}, \left(\frac{1}{m}\right)$$

$$2. \gamma = \log_e \left(\frac{I_s}{I_L} \right)$$

$$= \log_e \left(\frac{V_s}{V_L} \right), \text{ nepers}$$

$$3. \gamma = 20 \log_{10} \left(\frac{I_s}{I_L} \right)$$

$$= 20 \log_{10} \left(\frac{V_s}{V_L} \right), \text{ dB}$$

4. $\gamma = \alpha + j\beta$

α = attenuation constant, dB/m

β = phase constant, rad/m.

It may be noted that,

$$1 \text{ neper} = 8.686 \text{ dB}$$

$$1 \text{ rad} = 57.3^\circ$$

* consider,

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$\gamma^2 = \alpha^2 + 2j\alpha\beta - \beta^2$$

$$= RG - \omega^2 LC + j\omega RC + j\omega LC$$

$$\alpha^2 - \beta^2 = RG - \omega^2 LC \longrightarrow (5)$$

$$|\gamma| = \sqrt{\alpha^2 + \beta^2}$$

$$\alpha^2 + \beta^2 = \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} \longrightarrow (6)$$

from equation's 5 and 6, we get by adding,

$$2\alpha^2 = (RG - \omega^2 LC) + \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}$$

$$\alpha = \sqrt{\frac{1}{2} \left[(RG - \omega^2 LC) + \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} \right]}$$

from equation's 5 and 6, by subtracting those eq'n's we get,

$$2\beta^2 = (RG - \omega^2 LC) - \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)}$$

$$\beta = \sqrt{\frac{1}{2} \left[(\omega^2 LC - RG) + \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} \right]}$$

Characteristic Impedance (Z_0):

1. It is defined as the ratio of forward voltage wave to the forward current wave at any point on the line,

$$Z_0 = \frac{V_f}{I_f}, \Omega$$

2. It is also defined as the ratio of the square root of series impedance to square root of shunt admittance

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

It can also be defined as the ratio of reflected voltage wave to the reflected current wave at any point on line.

$$Z_0 = -\frac{V_r}{I_r}$$

Lossless Transmission Lines

A transmission line is said to be lossless if the conductors of line are perfect, or $\sigma_c = \infty$ and the dielectric medium between the lines is lossless, or, $\sigma_d = 0$. Also, a line is said to be lossless, if

$$R = 0 = G$$

For lossless line,

$$Y = \alpha + j\beta$$

$$= j\beta ; \alpha = 0$$

$$= \sqrt{(R + j\omega L)(G + j\omega C)}$$

As, $R = 0, G = 0$.

$$Y = j\omega \sqrt{LC} = j\beta$$

then

$$\beta = \omega \sqrt{LC}, \alpha = 0 \text{ for lossless line}$$

As $R = 0 = G$

$$Z_0 = \sqrt{\frac{L}{C}}$$

The velocity of propagation in lossless line

$$V_p = \frac{\omega}{\beta} = \frac{\omega}{\omega \sqrt{LC}}$$

$$V_p = \frac{1}{\sqrt{LC}}$$

Distortionless Line

A transmission line is said to be distortionless when the attenuation constant, α is frequency-independent and the phase constant, β is linearly dependent on the frequency or when.

$$\frac{R}{L} = \frac{G}{C}$$

consider,

$$Y = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$= \sqrt{RG \left(1 + \frac{j\omega L}{R}\right) \left(1 + \frac{j\omega C}{G}\right)}$$

if, $\frac{R}{L} = \frac{G}{C}$

$$= \sqrt{RG} \left(1 + \frac{j\omega C}{G}\right)$$

$$\alpha = \sqrt{RG}, \quad \beta = \omega \sqrt{LC}$$

consider,

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$$= \sqrt{\frac{R \left(1 + \frac{j\omega L}{R}\right)}{G \left(1 + \frac{j\omega C}{G}\right)}}$$

$$= \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}} = R_0 + jX_0. \text{ that is } X_0 = 0.$$

$$Z_0 = R_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}} \text{ for distortion less line}$$

The velocity of propagation for distortion less is

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega \sqrt{LC}} = \frac{1}{\sqrt{LC}}$$

The overall transmission line characteristics

Parameter	General transmission line	Lossless line	Distortion less line
Y	$\sqrt{(R + j\omega L)(G + j\omega C)}$	$j\omega \sqrt{LC}$	$\sqrt{RG} + j\omega \sqrt{LC}$
Z_0	$\sqrt{\frac{R + j\omega L}{G + j\omega C}}$	$\sqrt{\frac{L}{C}}$	$\sqrt{\frac{L}{C}}$ or $\sqrt{\frac{R}{G}}$
α	$\left[\frac{1}{2}(RG - \omega^2 LC) + \frac{1}{2} \sqrt{(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)} \right]^{\frac{1}{2}}$	0	$\sqrt{RG}$

Parameter	General transmission line	lossless line	Distortion less line
β	$\left[\frac{1}{2}(\omega^2 LC - RG) + \frac{1}{2}[(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)]^{1/2} \right]^{1/2}$	$\omega \sqrt{LC}$	$\omega \sqrt{LC}$
v_p	$\frac{\omega}{\beta}$	$\frac{1}{\sqrt{LC}}$	$\frac{1}{\sqrt{LC}}$

Phase And Group Velocities:-

Energy is propagated along a transmission line in the form of Transverse electromagnetic wave. The phase velocity, v_p for TEM wave is

$$v_p = \frac{1}{\sqrt{\mu\epsilon}}$$

for a transmission line, $\mu_r = 1$, but ϵ_r may be different. Then

$$v_p = \frac{v_0}{\sqrt{\epsilon_r}}$$

v_0 = free space velocity.

for distortionless transmission line,

$$v_p = \frac{1}{\sqrt{LC}}$$

Individual wave propagate with the same phase velocity, if β is proportional to ω . If β is not proportional to ω and if wave components travel with different velocities, the envelope of wave travels with a velocity, known as group velocity, v_g

$$v_g = \frac{\partial \omega}{\partial \beta}$$

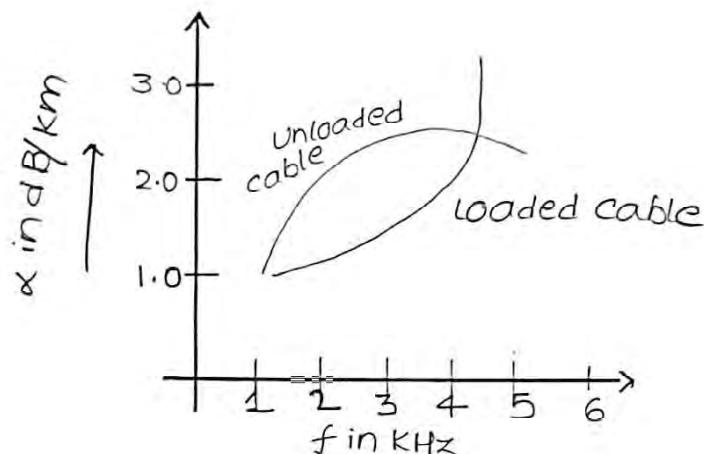
This is also the velocity at which energy is propagated

along the line.

Loading of Lines:-

Introduction of inductance in series with the line is called loading and such lines are called "loaded lines".

Effect of Loading



Types of Loading

1. Continuous Loading Here loading is done by winding a type of iron around the conductor. This increases inductance but it is expensive.
2. Patch Loading This type of loading uses sections of continuously loaded cable separated by unloaded cable. Hence cost is reduced.
3. Lumped Loading Hence, loading is introduced at uniform intervals. It may be noted that hysteresis and eddy current losses are introduced by loading and hence design should be optimal.

Input Impedance of lossless Transmission Line for lossless line,

$$Y = jB \quad [\text{as } \alpha = 0]$$

and as $\tanh j\beta l = j \tan \beta l$ and $Z_0 = R_0$, Z_i becomes

$$Z_i = R_0 \left[\frac{Z_L + jR_0 \tan \beta l}{R_0 + jZ_L \tan \beta l} \right] \text{ for a lossless line,}$$

for shorted line, $Z_L = 0$

$$Z_{sc} = Z_0 \tanh \gamma L$$

for open circuited line, $Z_L = \infty$

$$Z_{oc} = Z_i = Z_0 \coth \gamma L$$

and

$$(Z_i)_{sc} (Z_i)_{oc} = Z_0^2$$

for matched line,

$$Z_L = Z_0$$

$$Z_i = Z_0$$

Input Impedance of a transmission line for different cases are given below.

Type of line	Input impedance, Z_i
Lossy line	$Z_0 \frac{Z_L + Z_0 \tanh \gamma L}{Z_0 + Z_L \tanh \gamma L}$
lossless line	$Z_0 \frac{Z_L + j Z_0 \tan \beta l}{Z_0 + j Z_L \tan \beta l}$
lossy line with shorted load ($Z_L = 0$)	$Z_0 \tanh \beta l$
lossy line with open circuited load ($Z_L = \infty$)	$Z_0 \coth \beta l$
lossy line with matched load ($Z_L = Z_0$)	Z_0
lossy line with shorted load	$j Z_0 \tan \beta l$
lossless line with open circuited load	$-j Z_0 \cot \beta l$
lossless line with matched load	Z_0

The variation of input impedance of lossless line are shown.

Electro Magnetic Waves
And
Transmission Lines

UNIT-5 . Transmission Lines - II

Topics of Transmission Lines - II

1. Input Impedance Relations
2. SC and OC Lines
3. Reflection co-efficient
4. Voltage standing wave ratio (VSWR)
5. Low loss radio frequency lines and UHF Transmission lines
6. UHF Lines as circuit Elements
7. Impedance transformations $\frac{\lambda}{4}$, $\frac{\lambda}{2}$, $\frac{\lambda}{8}$ lines
8. Smith chart construction & Applications
9. Quarter wave transformer
10. stub Matching - single & double, Illustrative problems

1. Input impedance relations:

Input impedance: At the entry point of a transmission line, signals encounter input impedance that limits the flow of current through it.

"The input impedance depends on the complete set of elements present in the circuit"

For lossless line,

$$\gamma = j\beta \quad [\text{as } \alpha = 0]$$

and as $\tanh j\beta l = j \tan \beta l$

$$Z_0 = R_0$$

Z_i becomes,

$$Z_i = R_0 \left[\frac{Z_L + j R_0 \tan \beta l}{R_0 + j Z_L \tan \beta l} \right] \quad \text{for a lossless line}$$

Z_L is the termination impedance

l is line length

$$\beta = 2\pi f / (v_p)$$

For shorted line, $Z_L = 0$

$$Z_{sc} = Z_0 \tanh j\beta l$$

For open circuited line, $Z_L = \infty$

$$Z_{oc} = Z_i = Z_0 \coth j\beta l$$

and

$$(Z_i)_{sc} (Z_i)_{oc} = Z_0^2$$

and for matched line,

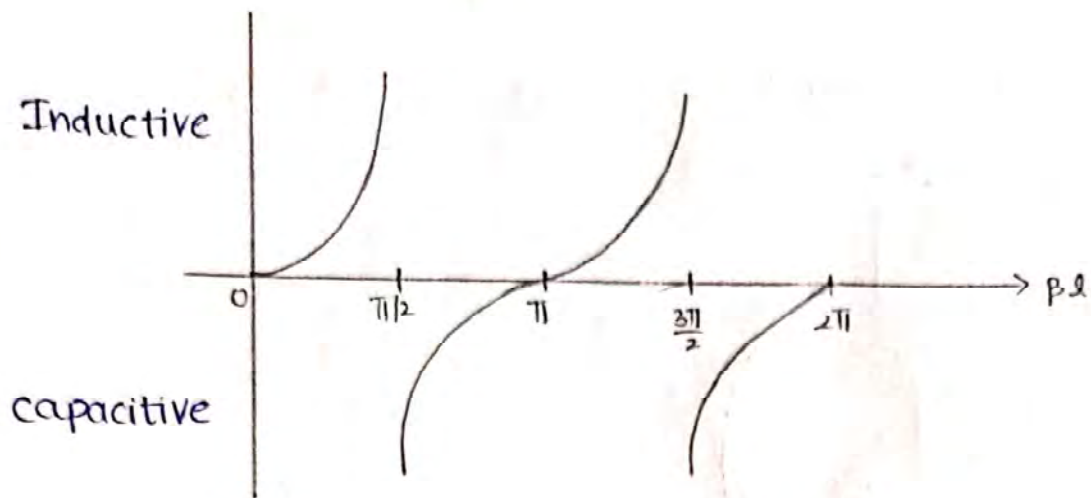
$$Z_L = Z_0$$

$$Z_i = Z_0$$

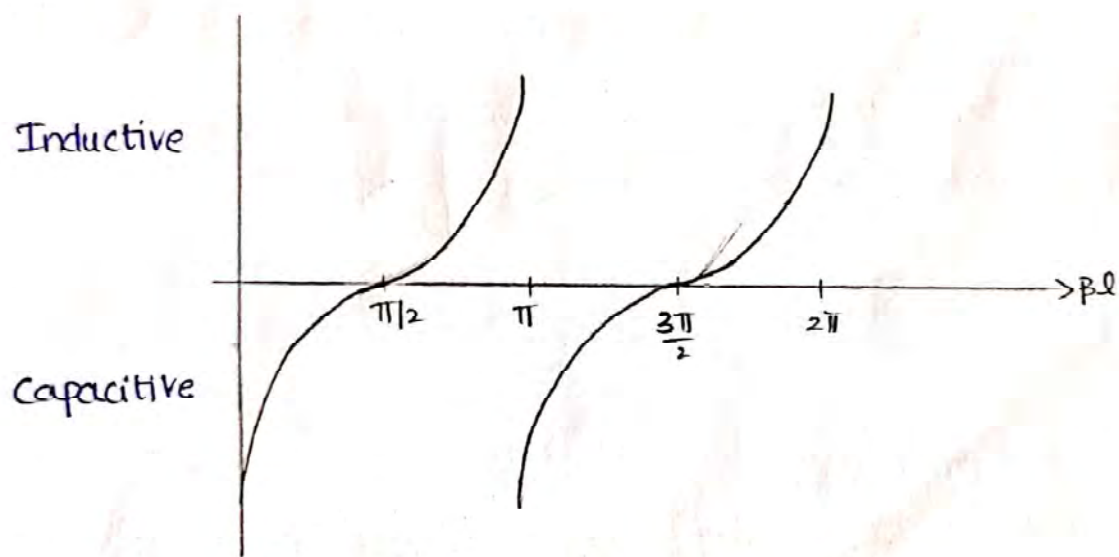
*Input impedances of a transmission line for different cases are given in the below table

Type of line	Input impedance, Z_i
Lossy line	$Z_0 \left[\frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l} \right]$
Lossless line	$Z_0 \left[\frac{Z_L + j Z_0 \tan \beta l}{Z_0 + j Z_L \tan \beta l} \right]$
Lossy line with shorted load ($Z_L = 0$)	$Z_0 \tanh \gamma l$
Lossy line with open circuited load ($Z_L = \infty$)	$Z_0 \coth \gamma l$
Lossy line with matched load ($Z_L = Z_0$)	Z_0
Loss less line with shorted load	$j Z_0 \tan \beta l$
Loss less line with open circuited load	$-j Z_0 \cot \beta l$
Loss less line with matched load	Z_0

The Variation of input impedance of lossless line when shorted and open circuited are shown below



a) short load.



b) open circuited load.

Open circuited and short circuited lines:

Open circuited Transmission Line:

When the transmission line is opened at the load end, it is known as open circuited line.

$$Z_R = \infty$$

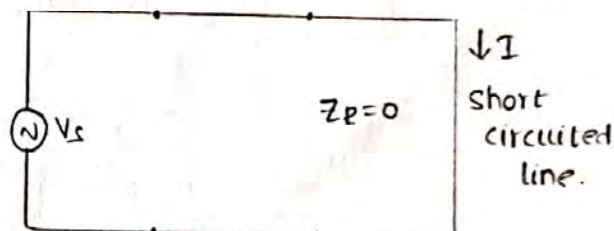


At open circuited end, the voltage is maximum and the current is minimum.

Short circuited Transmission Line:

When the transmission line is short circuited at the load end, the line is known as short circuited line.

$$Z_R = 0$$



At short circuited end, the current is maximum and the voltage is minimum.

Reflection co-efficient:

The Reflection coefficient is a parameter that describes how much of a wave is reflected by an impedance discontinuity in the transmission line. It is equal to the ratio of the amplitude of the reflected wave to the incident wave, with each expressed as phasors.

$$\rho = \frac{Z_L - Z_0}{Z_L + Z_0}$$

* Reflect coefficient is also defined as the ratio of difference of load impedance and characteristic impedance to the sum of the load impedance and characteristic impedance.

where ρ = reflection co-efficient

Z_L = load impedance

Z_0 = characteristic impedance

Relation between Reflection coefficient, Load and the Characteristic Impedances:

Proof:

For lossless RF lines, we have

$$\gamma = j\beta$$

$$V_{\text{incident}} = V_i e^{j\beta l}$$

Similarly,

$$V_{\text{reflected}} = V_r e^{-j\beta l}$$

At the load, ($l=0$)

$$V_L = V_i + V_r$$

$$Z_0 = \frac{V_{\text{incident}}}{I_{\text{incident}}} = \frac{-V_{\text{reflected}}}{I_{\text{reflected}}}$$

I at any point from load is

$$I = I_{\text{incident}} + I_{\text{reflected}}$$

Load current, I_L

$$I_L = I_{\text{incident}} - I_{\text{reflected}}$$

$$= \frac{V_{\text{incident}}}{Z_0} - \frac{V_{\text{reflected}}}{Z_0}$$

$$I_L = \frac{V_i - V_r}{Z_0}$$

But by definition, the reflection coefficient, ρ is

$$\rho = \frac{V_r}{V_i}$$

and

$$Z_L = \frac{V_L}{I_L} = \frac{(V_i + V_r)}{(V_i - V_r)} Z_0$$

$$V_i Z_L - Z_L V_r = Z_0 V_i + Z_0 V_r$$

Dividing both sides by V_i , we get

$$Z_L - Z_L \frac{V_r}{V_i} = Z_0 + Z_0 \frac{V_r}{V_i}$$

$$\rho = \frac{V_r}{V_i} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Hence proved.

$$\text{If } Z_L = Z_0, \quad \rho = 0$$

$$Z_L = 0, \text{ then } \rho = -1$$

$$Z_L = \infty, \text{ then } \rho = 1$$

$$Z_L = \text{purely reactive, then } |\rho| = 1$$

Voltage Standing Wave Ratio (VSWR)

Voltage standing wave ratio (VSWR) is defined as the ratio between transmitted and reflected voltage. Standing waves in radio frequency (RF) electrical transmission system. It is a measure of how efficiently RF power is transmitted from power source, through a transmission line, and into the load.

* A common example is a power amplifier connected to an antenna through a transmission line.

Relation between Reflection coefficient and Voltage Standing wave Ratio (VSWR) (S)

VSWR is defined as

$$VSWR = \frac{V_{max}}{V_{min}}$$

Now VSWR is expressed in terms of Reflection coefficient

$$VSWR = \frac{1 + |P|}{1 - |P|}$$

Proof: we can write

$$\begin{aligned} V_{max} &= |V_i| + |V_r| \\ &= |V_i| [1 + |P|] \end{aligned}$$

Similarly,

$$V_{min} = |V_i| - |V_r| = |V_i| (1 - |P|)$$

$$VSWR = S = \frac{|V_i| (1 + |P|)}{|V_i| (1 - |P|)}$$

$$VSWR = \frac{1 + |P|}{1 - |P|}$$

Hence Proved

S ranges between 1 and ∞ or $1 < S < \infty$

we can also write |P| as

$$|P| = \frac{S - 1}{S + 1}$$

when the line is terminated by purely resistive load,

$$S = \frac{Z_0}{R_L} \quad \text{if } Z_0 > R_L$$

$$S = \frac{R_L}{Z_0} \quad \text{if } R_L > Z_0$$

Low loss radio Frequency lines & UHF Transmission lines

Radio Frequency Lines (RF lines)

Radio frequency (RF) is the oscillation rate of an alternating electric current or voltage or of magnetic, electric or electromagnetic field or mechanical system in the frequency range from around 20 KHz to around 300 GHz. This is roughly between the upper limit of audio-frequencies and lower limit of infrared-frequencies

At radio-frequencies,

$$\omega L \gg R$$

$$\omega C \gg G$$

Then,

$$Z = R + j\omega L \approx j\omega L$$

$$Y = G + j\omega C \approx j\omega C$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \approx \sqrt{\frac{L}{C}}$$

and

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$\gamma \approx j\omega\sqrt{LC}$$

As, $\gamma = \alpha + j\beta$, $\alpha \approx 0$, $\beta \approx \omega\sqrt{LC}$, it is not enough if α is small compared to β .

Hence let us consider,

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$= j\omega\sqrt{LC} \sqrt{\left(1 + \frac{R}{j\omega L}\right)\left(1 + \frac{G}{j\omega C}\right)}$$

$$\approx j\omega\sqrt{LC} \left(1 + \frac{R}{2j\omega L}\right)\left(1 + \frac{G}{2j\omega C}\right)$$

$$= \frac{R}{2\sqrt{L/C}} + \frac{G}{2} \sqrt{L/C} + j\omega\sqrt{LC}$$

$$\alpha \approx \frac{1}{2} \left(\frac{R}{Z_0} + G Z_0 \right)$$

$$\beta = \omega\sqrt{LC}$$

For RF lines, the input impedance is

$$Z_i = Z_0 \left[\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right]$$

Ultra High Frequency Lines (UHF lines)

The transmission lines that are operated at a frequency range from 300MHz to 3GHz are known as Ultra-High Frequency (UHF) lines. At low frequencies or under normal frequencies the transmission lines are used as wave guides for transferring power and information from one point to another.

At Ultra-High Frequencies, the transmission lines can be used as circuit elements like capacitor or an inductor. It means they can be used in circuits like a Capacitor or an inductor.

UHF lines as circuit elements:

At low frequencies, circuit elements are lumped since the voltage and current waves affect the entire circuit at the same time. whereas UHF, such treatment is not possible, since voltage and current waves do not affect the entire circuit at the same time.

This is mainly because of the length (l) of the circuits are much greater than the wavelengths (i.e. $l \gg \lambda$)

1. If a line of length $< \lambda/4$ is terminated with short circuit, then it is $\cong$ inductance. If a line of length $< \lambda/4$ is terminated with open circuit, then $\cong$ capacitance.
2. If a line of length is between $\lambda/4$ and $\lambda/2$ is terminated with short circuit, then it is $\cong$ capacitance.
3. If a line of length is between $\lambda/4$ and $\lambda/2$ is terminated with open circuit, then it is $\cong$ inductance.
4. If a line of length equal to $\lambda/4$ is terminated with short circuit, then it is $\cong$ parallel resonant circuit.
5. If a line of length equal to $\lambda/4$ is terminated with open circuit, then it is $\cong$ series resonant circuit.

* Note: The line impedance repeats for successive increase of length $\lambda/4$.

Impedance Transformations $\frac{\lambda}{4}, \frac{\lambda}{2}, \frac{\lambda}{8}$ lines

For quarter wavelength ($\lambda/4$) transmission line with short circuit in terms of R and Z_0 and length of the line (l) the input impedance of short circuited line is given by

$$Z_{in|s.c} = Z_0 \tanh(\gamma l)$$

$$\Rightarrow Z_0 \left[\frac{\sinh(\gamma l)}{\cosh(\gamma l)} \right] = Z_0 \left[\frac{\sinh(\alpha l) \cos \beta l + j \cosh(\alpha l) \sin \beta l}{\cosh(\alpha l) \cos \beta l + j \sinh(\alpha l) \sin \beta l} \right]$$

For line of lengths equal to an odd multiples of $\lambda/4$,

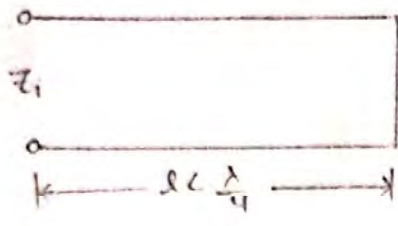
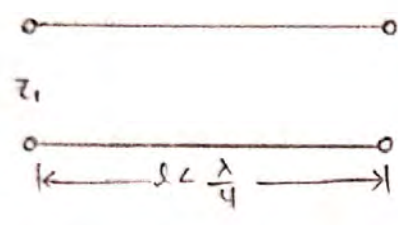
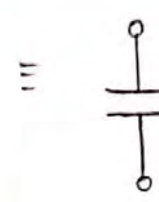
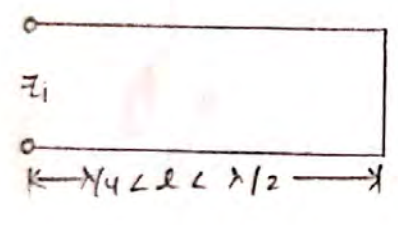
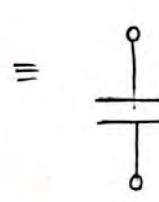
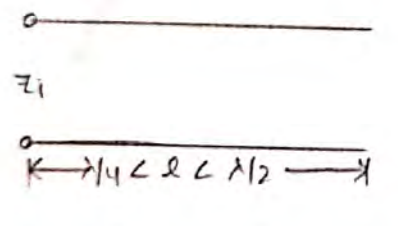
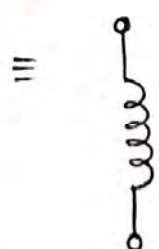
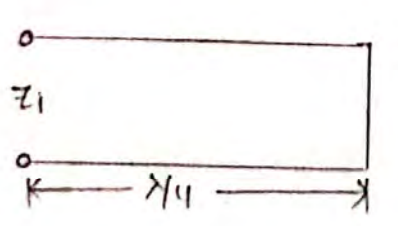
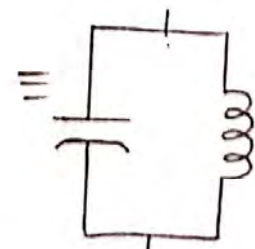
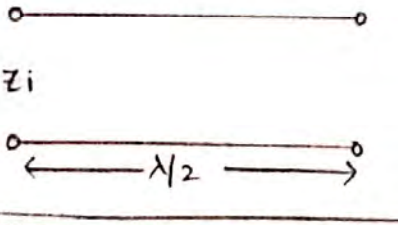
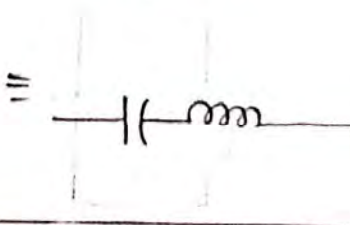
$$\sin \beta l = \pm 1 \quad \text{and} \quad \cos \beta l = 0.$$

$$\therefore Z_{in|s.c} = Z_0 \left[\frac{\cosh(\alpha l)}{\sinh(\alpha l)} \right]$$

But when we are working at high frequencies, dl is small and so $\cosh dl = 1$ and $\sinh dl = dl$.

$$\therefore Z_{in|s.c} = Z_0 \left[\frac{1}{(dl)} \right] \quad \text{--- (1)}$$

Equivalent circuit and Impedance

Transmission line	Equivalent circuit	Input impedance
		$Z_i = +jZ_0 \tan \beta l$
		$Z_i = -jZ_0 \cot \beta l$
		$Z_i = +jZ_0 \tan \beta l$
		$Z_i = -jZ_0 \cot \beta l$
		$Z_i \approx \frac{2Z_0^2}{Rl} = \frac{Z_0^2}{\tanh \beta l}$
		$Z_i \approx \frac{2Z_0^2}{Rl} = \frac{Z_0^2}{\tanh \beta l}$

We know that

$$\alpha \cong \frac{1}{2} \left(R \sqrt{\frac{C}{L}} + G \sqrt{\frac{L}{C}} \right)$$

The loss due to conductance (G) can be neglected in case of air, dielectric lines. so,

$$\alpha \cong \frac{1}{2} \left(R \sqrt{\frac{C}{L}} \right)$$

where

$$Z_0 = \sqrt{L/C}$$

By substituting "d", we get

$$Z_{in|s.c} = \left[\frac{2Z_0^2}{(Rl)} \right]$$

∴ Input impedance for quarter wavelength transmission line terminated with short circuit is

$$Z_{in|s.c} = \left[\frac{2Z_0^2}{(Rl)} \right]$$

Half - wavelength ($\lambda/2$) Line impedance Transformation

consider a line of length l terminated with a load Z_L , for the half wavelength ($\lambda/2$) transmission line section

$$l = \lambda/2$$

$$\Rightarrow \beta l = \left(\frac{2\pi}{\lambda} \right) \left(\frac{\lambda}{2} \right) = \pi$$

$$\Rightarrow \tan(\beta l) = 0$$

The input impedance for the half wavelength section is

$$Z_{in} = Z_0 \left[\frac{Z_L + Z_0 j \tan(\beta l)}{Z_0 + Z_L j \tan(\beta l)} \right]$$

$$Z_{in} = Z_L$$

This implies that the input impedance of the $\lambda/2$ line is same as the load impedance of that line.

Quarter-wavelength ($\lambda/4$) Line Impedance Transformation

For the quarter wavelength transmission line section

$$l = \frac{\lambda}{4}$$

$$\Rightarrow \beta l = \left(\frac{2\pi}{\lambda}\right) \left(\frac{\lambda}{4}\right) = \frac{\pi}{2}$$

$$\Rightarrow \tan(\beta l) = \infty$$

The input impedance for the quarter-wavelength section is

$$Z_{in} = Z_0 \left[\frac{Z_L + Z_0 j \tan(\beta l)}{Z_0 + Z_L j \tan(\beta l)} \right]$$

$$\Rightarrow Z_{in} = \frac{Z_0^2}{Z_L}$$

$$Z_0^2 = Z_{in} Z_L$$

$$Z_0 = \sqrt{Z_{in} Z_L}$$

if $Z_L = \infty$, then $Z_{in} = 0$

Hence, a line of length $\lambda/4$ transforms high impedance to low impedance and viceversa.

$\lambda/8$ Line Impedance Transformation

$$l = \frac{\lambda}{8}$$

$$\Rightarrow \beta l = \left(\frac{2\pi}{\lambda}\right) \left(\frac{\lambda}{8}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan \beta l = 1$$

The input impedance for the $\lambda/8$ line section is:

$$Z_{in} = Z_0 \left[\frac{Z_L + j Z_0 \tan(\beta l)}{Z_0 + j Z_L \tan(\beta l)} \right]$$

$$\Rightarrow Z_{in} = Z_0 \left[\frac{Z_L + jZ_0}{Z_0 + jZ_L} \right]$$

$$\Rightarrow |Z_{in}| = |Z_0|$$

For a $\lambda/8$ transmission line, the magnitude of input impedance is the same as the magnitude of characteristic impedance.

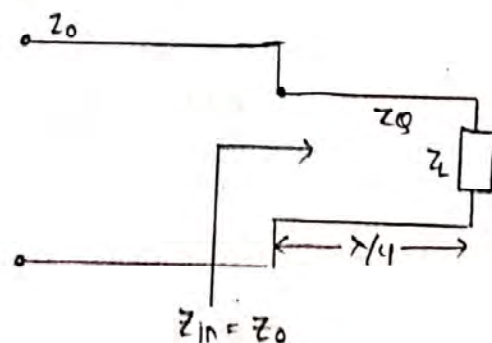
Impedance Matching

There are various techniques in order to achieve impedance matching. some are as follows

1. Quarter-wave transformer
2. Single stub
3. Double stub

Quarter-wave transformer

Reflection exist when the impedance mismatch occurs, that is, $Z_L \neq Z_0$ on the transmission line. A simple method for matching load impedance to transmission line is the use of a quarter-wave transformer that is a piece of transmission line having a $\lambda/4$ length and a characteristic impedance of Z_0 .



Derivation of Z_0 : Inserted line length

$$l = \frac{\lambda}{4} \Rightarrow \beta l = \left(\frac{2\pi}{\lambda} \right) \left(\frac{\lambda}{4} \right) = \frac{\pi}{2}$$

$$\Rightarrow \tan(\beta l) = \infty$$

The input impedance of quarter wave transformer with the load Z_L is

$$Z_{in} = Z_0 \left[\frac{Z_L + Z_0 j \tan(\beta l)}{Z_0 + Z_L j \tan(\beta l)} \right]$$

$$\Rightarrow Z_{in} = \frac{Z_0^2}{Z_L}$$

$$\Rightarrow Z_0^2 = Z_{in} Z_L$$

$$\Rightarrow Z_0 = \sqrt{Z_{in} Z_L}$$

Single and Double stub Matching

A small section of transmission line is known as a "stub". The transmission line realizing the stub is normally terminated by a short or an open circuit.

A short-circuited stub is less prone to the leakage of electromagnetic radiation when compared with an open-circuited stub. Since the circuit is based on the insertion of a parallel stub, it is more convenient to work with admittances rather than impedances.

Single stub:

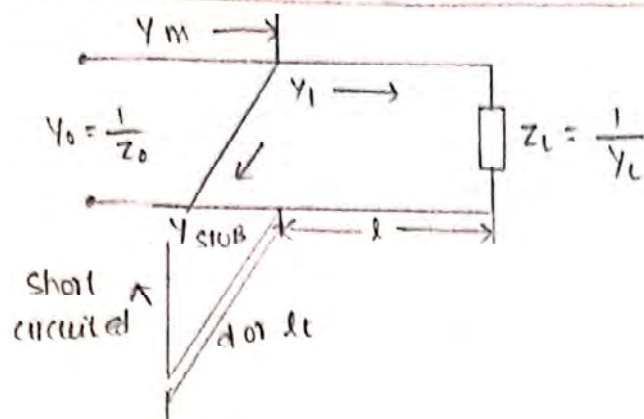
Any load impedance can be matched to the line by using the single stub technique.

The stub consists of a short or an open section of the transmission line of length d connected in parallel with the main line at some distance l from load.

There are two design parameters for single stub matching.

1. The location of the stub with reference to the load.

2. The length of the stub line d .



The location of stub (l) with reference to load is given by

$$l = \frac{1}{\beta} \tan^{-1} \left(\sqrt{\frac{Y_0}{Y_L}} \right) = \frac{1}{\beta} \tan^{-1} \left(\sqrt{\frac{Z_L}{Z_0}} \right)$$

The length of the stub with reference to load is given by

$$d = \frac{1}{\beta} \cot^{-1} \left(\sqrt{\frac{Z_0}{Z_L}} \left(1 - \frac{Z_0}{Z_L} \right) \right)$$

Positional length for a single stub

The positional length (l) for a single stub is

$$l = \frac{1}{\beta} \tan^{-1} \left(\sqrt{\frac{Y_0}{Y_L}} \right) = \frac{1}{\beta} \tan^{-1} \left(\sqrt{\frac{Z_L}{Z_0}} \right)$$

Proof:

$$Z_{in} = Z_0 \left(\frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l} \right)$$

$$Y_{in} = \frac{1}{Z_{in}}$$

$$= \frac{(Z_0 + Z_L \tanh \gamma l)}{Z_0 (Z_L + Z_0 \tanh \gamma l)}$$

$$= Y_0 \left[\frac{Y_L + Y_0 \tanh \gamma l}{Y_0 + Y_L \tanh \gamma l} \right]$$

Normalized input admittance is given by

$$Y_s = \frac{Y_{in}}{Y_0}$$

Substituting the value of Y_{in} we get

$$Y_s = \frac{Y_L / Y_0 + \tanh \gamma l}{1 + (Y_L / Y_0) \tanh \gamma l}$$

Also normalized load admittance

$$Y_R = \frac{Y_L}{Y_0}$$

Let $\gamma = j\beta$ for a lossless line, then

$$Y_s = \frac{Y_R + j \tan \beta l}{1 + j Y_R \tan \beta l}$$

Note:

The expression for the position of stub line " l_s " in terms of the reflection coefficient is

$$l_s = \frac{\lambda}{2\pi} [\phi + \pi - \cos^{-1}(|\Gamma|)]$$

Length of the stub (short circuit) for a single stub

Length (d) of the stub (short circuit) for a single stub is given by

$$d = \frac{1}{\beta} \cot^{-1} \left(\sqrt{\frac{Z_L}{Z_0}} \left(1 - \frac{Z_0}{Z_L} \right) \right)$$

Advantages of single stub:

1. It is mostly used for fixed impedances at microwave frequencies.
2. The length and characteristic impedance need not be changed.
3. Mechanical adjustment of the length of the stub and the position of the stub is possible for matching.

Disadvantages of single stub:

The main drawback is that if frequency changes, the location of stub has to be changed, which is not possible in single stub.

Double stub:

To overcome the disadvantages of single stub, there is another possible method of impedance matching known as double stub matching. It uses two short circuited stubs whose lengths are adjustable independently according to the requirements and the spacing is frequently made $\lambda/4$ but whose positions are fixed.

In double stub matching, two stubs are separated by length $\lambda/4$. By using two stubs, one can achieve a match for a wide range of loads by adjusting the lengths of the stubs.

Smith chart construction and Applications:

Smith chart has been developed to reduce the tedious manipulations involved in calculating the characteristic of the transmission lines. It is used to compute the transmission line input impedance, admittance, reflection coefficient and VSWR without using formula. In this, the resistive components R and reactive component X are represented in a circular form. The resistive components and the reactive components each form independent set of circles. Superposing these two sets of circles gives the smith chart.

Smith chart configuration:

Smith chart is one of the most precious graphical tools in the electromagnetic theory. It is very useful in simplifying the complex problems in transmission lines. This chart is reflection coefficient to impedance converter or impedance to reflection coefficient converter.

The reflection coefficient is defined as

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\Rightarrow \Gamma_L = \frac{Z_L / Z_0 - 1}{Z_L / Z_0 + 1}$$

$$\Rightarrow \Gamma_L = \frac{Z_N - 1}{Z_N + 1}$$

where $Z_N = Z_L / Z_0$ is the normalized impedance and can be expressed as $r + jx$ and Z_0 is the characteristic impedance of transmission line

$$\Gamma_L = \frac{r + jx - 1}{r + jx + 1} = \text{Re}\{\Gamma_L\} + j\text{Im}\{\Gamma_L\} = A + jB \quad \text{--- ①}$$

Equating real and imaginary parts in above equation, we get

$$\text{Re}\{\Gamma_L\} = A = \frac{r^2 - 1 + x^2}{(r+1)^2 + x^2} \quad \text{--- ②}$$

$$\text{Im}\{\Gamma_L\} = B = \frac{2x}{(r+1)^2 + x^2} \quad \text{--- ③}$$

from above two equations, the following two families of circles are obtained.

1. constant- r circles

2. constant- x circles

1. Constant- r circles

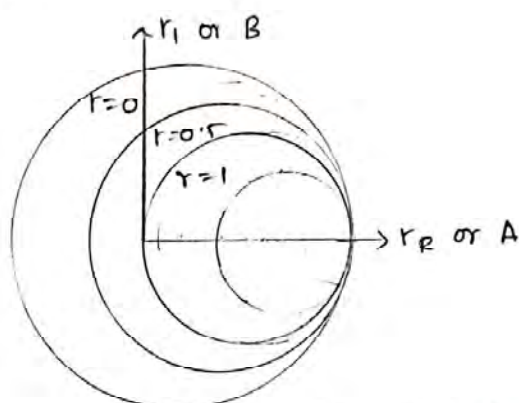
These circles are obtained from eqns ② & ③ by eliminating x :

$$\left[A - \left(\frac{r}{r+1} \right) \right]^2 + B^2 = \left(\frac{1}{r+1} \right)^2$$

Equation (4) represents a circle with a Radius $R = 1/(1+r)$ and centered at $(A_0, B_0) = (r/(1+r), 0)$.

The circles have the following features

1. The centers of all the r -circles lie on horizontal axis
2. From the analysis
 - when r is zero, then the circle is the outermost circle
 - in the smith chart, the center is at origin and radius is 1 unit
 - when r is ∞ , then the circle becomes a point $(1, 0)$
3. As the value of r increases from 0 to ∞ , the circles become progressively smaller.
4. All r -circles pass through the point $A=1, B=0$.

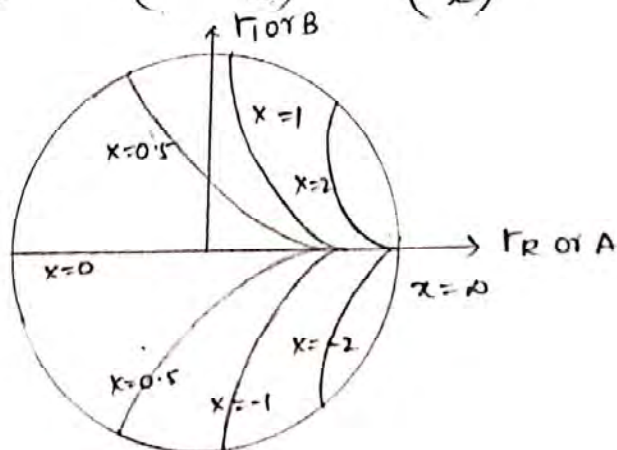


Constant- r circles as shown on r_R and r_I plane

Constant- x circles:

The second family of circles is obtained by eliminating r

$$(A-1)^2 + \left(B - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$$



The x -circles have the following features

1. The centers of all the x -circles lie on $A=1$ line.
2. The circles with $x > 0$ (inductive reactance) lie above the A -axis and the circles $x < 0$ (capacitive reactance) lie below the A -axis as shown.
3. As the value of $|x|$ increases from $x=0$ to $|x|=\infty$, the circle becomes progressively smaller.
4. The normalized reactance $x = \pm \infty$ is at the point $A=1, B=0$.
5. All constant x -circles are centered on a shifted line parallel to imaginary axis.

Applications of Smith chart:

Smith chart is used in the analysis of distributed elements (transmission lines) and lumped elements

1. Distributed elements

- Input impedance of a transmission line terminated with the load impedance.
- Reflection coefficient
- Plotting of standing wave pattern.
- Locations where the stubs are to be inserted
- SWR

2. Lumped Elements

- Input impedance for series and parallel lumped elements
- Frequency response of RLC circuits

1. Find the following parameters from Smith chart.

1. VSWR

2. Reflection co-efficient

3. Input impedance, if $Z_0 = 50\Omega$, $Z_L = 100 + j40\Omega$ and length $= 0.25\lambda$

Ans:

Step 1: Find Normalized load impedance

$$Z_L = \frac{Z_L}{Z_0} = \frac{100 + j40}{50} = 2 + j0.8$$

Step 2: Draw VSWR circle

Identify the VSWR circle and imaginary circle.

Step 3: Draw a straight line from center to load impedance

Step 4: wavelength $\lambda = \frac{c}{f} = 10\text{cm}$,

so length $L = 0.25\lambda$, Normalized input impedance $= 0.43 - j0.17$

Step 5: Find the point of intersection with VSWR circle at 0.25λ towards Generator

Z_0 .

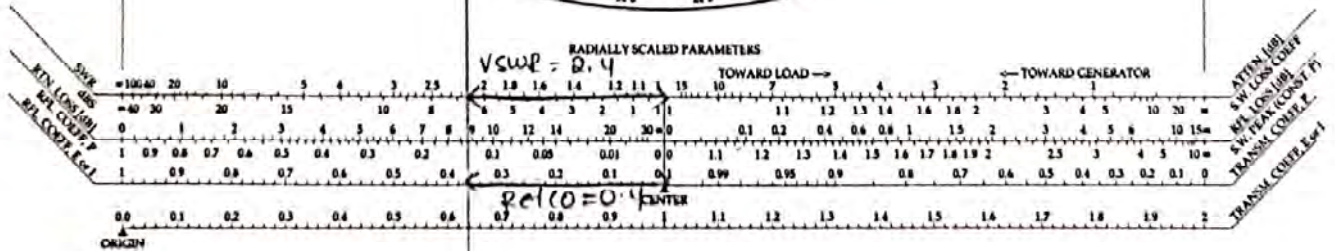
Actual input impedance = Normalized input impedance $\times$
characteristic impedance

$$= (0.43 - j0.17) 50$$

$$= 21.5 - j8.5\Omega$$

①

Smith Chart



Name:

Roll Number:

2. Find length and position of stub in series stub matching impedance whereas $Z_L = 0.4 - j0.4$

Sol: Given $Z_L = 0.4 - j0.4$

Step 1: Draw VSWR circle from load impedance Z_L .

Step 2: Draw unit real circle based on given data.

Step 3: Find the intersection to unit circle from Z_L in clockwise direction to get Z_1 .

Step 4: Identify the intersection point with the help of VSWR circle and unit circle. $\therefore Z_1 = 1 + j1.3$

Step 5: Find the Z_2 based on the following condition for impedance matching i.e. $Z_1 + Z_2 = 1$

$$\begin{aligned}\text{Step 6: Find } Z_2 \text{ from } Z_1, \text{ i.e. } Z_2 &= 1 - Z_1 \\ &= 1 - (1 + j1.3) \\ &= 1 - 1 - j1.3 \\ Z_2 &= -j1.3\end{aligned}$$

Step 7: Note Z_{sc} on Smith chart

Step 8: Find position of stub l_1 , which is from Z_L to Z_1 in clockwise direction.

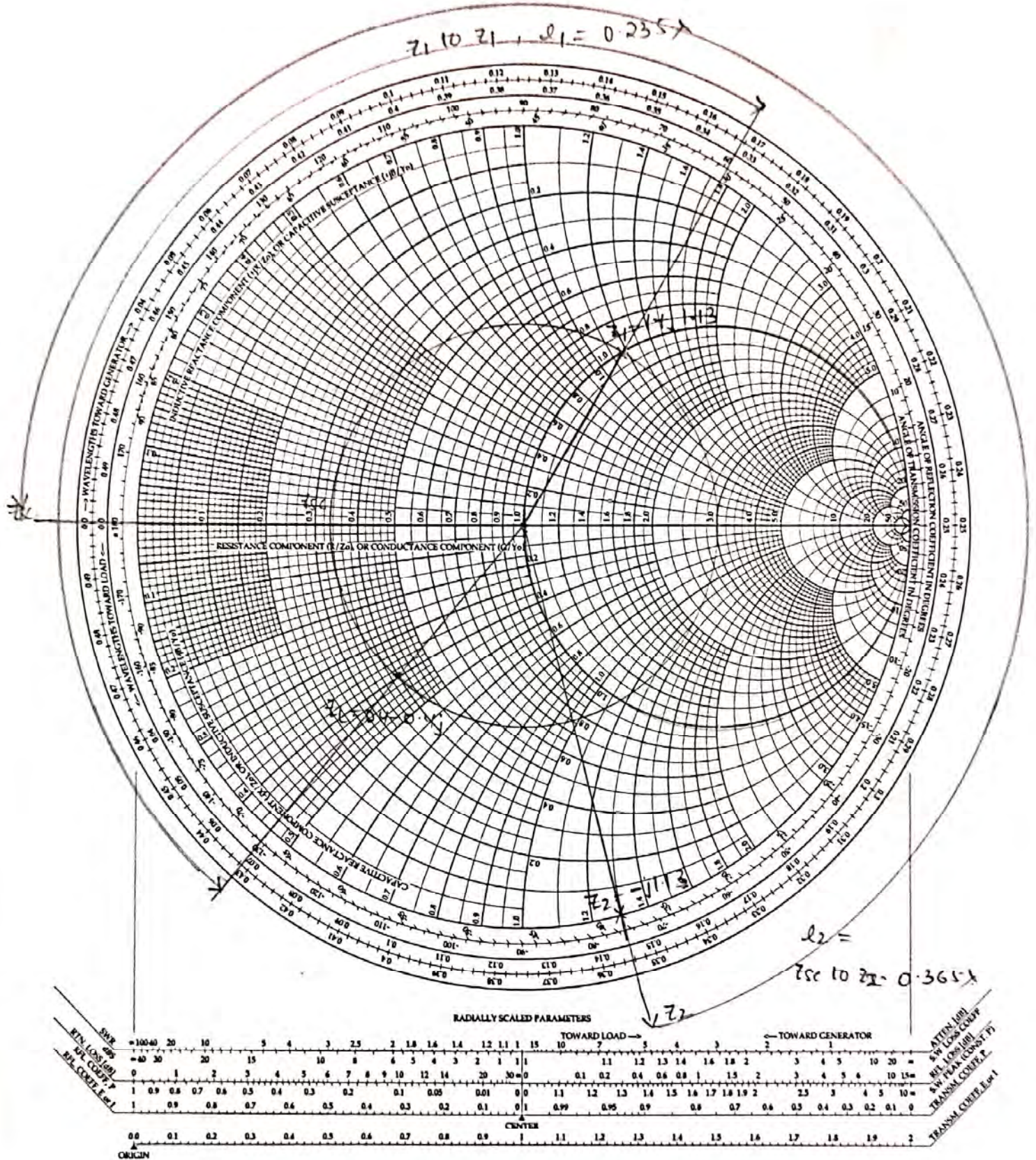
Step 9: Find length of stub l_2 , which is from Z_{sc} to Z_2 in clockwise direction.

$$\therefore l_1 = 0.6235 \lambda$$

$$l_2 = 0.365 \lambda$$

② 0.4-10.4

Smith Chart



Name:

Roll Number: